Pearson Edexcel

INTERNATIONAL ADVANCED LEVEL

# MATHEMATICS/ FURTHER MATHEMATICS/ PURE MATHEMATICS

# SCHEME OF WORK

# Mechanics 3

Pearson Edexcel International Advanced Subsidiary in Mathematics (XMA01)
Pearson Edexcel International Advanced Subsidiary in Further Mathematics (XFM01)
Pearson Edexcel International Advanced Subsidiary in Pure Mathematics (XPM01)
Pearson Edexcel International Advanced Level in Mathematics (YMA01)
Pearson Edexcel International Advanced Level in Further Mathematics (YFM01) Pearson Edexcel International Advanced Level in Pure Mathematics (YPM01)
First teaching September 2018
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# A level Further Mathematics: Mechanics 3

## Mechanics 3

| Unit | | Title | Estimated hours |
|---|---|---|---|
| **1** | | **Further kinematics:** Motion in a straight line when the acceleration is a function of the displacement ($x$) or time ($t$); Setting up and solving differential equations | **6** |
| **2** | | **Elastic strings and springs and elastic energy** | |
| | a | Hooke's law and definition of modulus of elasticity. Derivation of elastic potential energy formula | **6** |
| | b | Problem solving: equilibrium and using the work–energy principle | **5** |
| **3** | | **Further dynamics** | |
| | a | Particle moving in straight line with variable applied force; Using $F = ma$ to set up differential equations and solving | **6** |
| | b | Newton's law of gravitation | **4** |
| | c | Simple harmonic motion | **5** |
| **4** | | **Motion in a circle** | |
| | a | Angular speed, central force, radial acceleration | **3** |
| | b | Uniform motion in a horizontal circle | **6** |
| | c | Motion in a vertical circle | **6** |
| **5** | | **Statics of rigid bodies** | |
| | a | Centre of mass of uniform rod, lamina, 3D rigid body using integration (and symmetry); Deriving formulae in formula book | **5** |
| | b | Centre of mass of composite bodies; Simple cases of equilibrium of rigid bodies. | **4** |
| | c | Conditions for toppling/sliding | **4** |
| | | | **60 hours** |

# A level Further Mathematics: Mechanics 3

## UNIT 1: Further kinematics

**Motion in a straight line when the acceleration is a function of the displacement ($x$) or time ($t$). Setting up and solving differential equations (1.1)**

**Teaching time**
6 hours

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### SPECIFICATION REFERENCES

**1.1** Kinematics of a particle moving in a straight line when the acceleration is a function of the displacement ($x$), or time ($t$)

### PRIOR KNOWLEDGE

IAL Mathematics – Pure content

- Differentiation (Unit 4 of the P1 SoW and Unit 4 of the P3 SoW)
- Integration (Unit 5 of the P1 SoW and Unit 5 of the P4 SoW)
- Differential equations (Unit 6 of the P4 SoW)

IAS Mathematics – Mechanics 2 content

- Calculus in kinematics for motion in a straight line (Unit 1 of the M2 SoW)

### KEYWORDS

Particle, straight line, $x$-axis, one dimension, origin, displacement, kinematics, calculus, distance, time, velocity, speed, acceleration, deceleration, retardation, function of, magnitude, initially, maximum, minimum, differential equation, boundary conditions, integration, differentiation, chain rule, arbitrary constant, positive/negative direction, instantaneous, instantaneous rest.

### NOTES

This topic revises and extends the ideas from IAL Mechanics 2 where acceleration as a function of time was considered. The specification states the solution of the resulting equations 'will be consistent with the level of calculus required in units P1, P2, P3 and P4'.

### OBJECTIVES

By the end of the unit, students should:

- understand that acceleration may be written as $\frac{\mathrm{d}v}{\mathrm{d}t}$ or $v\frac{\mathrm{d}v}{\mathrm{d}x}$;
- be able to set up a first order differential equation in $v$ and $t$, or in $v$ and $x$, for acceleration in terms of $t$ or $x$;
- be able to set up a first order differential equation in $x$ and $t$ for velocity in terms of $t$ or $x$;
- be able to solve the resulting differential equation using calculus;
- be able to apply boundary/initial conditions to find arbitrary constants.

# A level Further Mathematics: Mechanics 3

## TEACHING POINTS

Give a general reminder that the *suvat* equations must only be used when the acceleration is constant; in practice, however, acceleration is often not constant.

Start with the familiar definition of acceleration as 'rate of change of velocity with respect to time' (with units of m $s^{-2}$ or 'metres per second per second').

In terms of calculus we write: $a = \frac{dv}{dt}$

Using the chain rule for differentiation we could write: $a = \frac{dv}{dx} \cdot \frac{dx}{dt}$

Now remember that velocity is defined as 'rate of change of displacement' which means we can also write $v = \frac{dx}{dt}$

So it follows that an alternative formula is: $a = v\frac{dv}{dx}$

Now consider $\frac{d(v^2)}{dx} = 2v\frac{dv}{dx}$ (using 'function of a function' differentiation) which leads to another formula: $a = \frac{1}{2}\frac{d(v^2)}{dx}$

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}$$

These formulae are key to problem solving in this unit.

If acceleration is given as f($t$), then we write $\frac{dv}{dt} = f(t)$ and integrate.

If acceleration is given as f($x$), then we write $v\frac{dv}{dx} = f(x)$, separate the variables and then integrate. Alternatively, write $\frac{1}{2}\frac{d(v^2)}{dx} = f(x)$ and integrate.

Start by looking at examples which give rise to straightforward integration. This will allow students to become familiar with the wording of questions and the application of boundary conditions. Remembering to include (and then to find) an arbitrary constant when integrating is essential, even if it subsequently turns out to be zero.

As in all of mechanics, remind students to be careful with signs. Displacement, velocity and acceleration are all vector quantities and, although we are only considering one dimensional motion in this section, care must be taken to include them as positive or negative within equations.

Displacement ($x$) is always measured from a fixed point (often the origin, $O$). Displacement, velocity and acceleration will be taken as positive in the direction of $x$ increasing. It is important to recognise that if the acceleration is given as 'towards $O$' then it should be written as a negative term.

If an expression for $v$ in terms of $t$ has been found or is given, integration should lead to an expression for the displacement by first writing $v = \frac{dx}{dt}$. Definite integration is useful for finding the distance travelled between two points, but care must be taken. Displacement and distance are *not* interchangeable; you can illustrate this using the following question.

# A level Further Mathematics: Mechanics 3

A particle $P$ moves on the positive $x$-axis.
The velocity of $P$ at time $t$ seconds is $(2t^2 - 9t + 4)$ m s$^{-1}$.
When $t = 0$, $P$ is 15 m from the origin $O$.
Find:

**(a)** the values of $t$ when $P$ is instantaneously at rest

**(b)** the acceleration of $P$ when $t = 5$

**(c)** the total distance travelled by $P$ in the interval $0 \leq t \leq 5$.

The answers to part **(a)** are $t = 0.5$ and $t = 4$ (put $v = 0$).

So what happens to the particle in the interval $0 \leq t \leq 5$?

At $t = 0$ the particle is travelling in positive $x$ direction. At $t = 0.5$ the particle reverses direction and then again at $t = 4$. This means that the total distance travelled is greater than the total displacement; hence integrating between limits 0 and 5 will not give the required answer.

Instead the integration has to be split and the separate distances combined:

$$\left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t\right]_0^{\frac{1}{2}} - \left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t\right]_{\frac{1}{2}}^{4} + \left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t\right]_4^5$$

which leads to a final answer of 19.4 m.

In tackling questions of this type it is a good idea to recommend sketching a velocity–time graph since it can be clearly seen when there is 'negative area'.

Any methods of integration from the IAL P1 to P4 content could be required in this topic. It may be a good idea to have a quick recap of some of these, in particular rational functions. Students should complete as wide a variety of questions as possible. It will have the added bonus of reinforcing their skills for the pure papers.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Below is an exam question which is fairly straightforward but which does lend itself to some discussion.

A particle $P$ moves on the positive $x$-axis.
When the distance of $P$ from the origin $O$ is $x$ metres, the acceleration of $P$ is $(7 - 2x)$ m s$^{-2}$, measured in the positive $x$-direction.
When $t = 0$, $P$ is at $O$ and is moving in the positive $x$-direction with speed 6 m s$^{-1}$.
Find the distance of $P$ from $O$ when $P$ first comes to instantaneous rest.

The acceleration is given in terms of $x$ so we choose the formula $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$.

We are also told that it is 'measured in positive $x$-direction' so the given expression goes in as positive.

So we write down either: $v\frac{\mathrm{d}v}{\mathrm{d}x} = 7 - 2x$

or: $\frac{1}{2}\frac{\mathrm{d}(v^2)}{\mathrm{d}x} = 7 - 2x$

which leads to: $\frac{1}{2}v^2 = 7x - x^2 + c$

Substituting initial condition when $x = 0$, $v = 6$ (note: speed given in positive direction so goes in as positive) leads to $c = 18$.

Discuss what 'instantaneous rest' actually means – why is it only instantaneous and what happens next?

Solving the quadratic equation (from $v = 0$) gives two values of $x$. Decide which is the correct solution here and discuss the subsequent motion of the particle together with the significance of the other solution. (The motion is actually SHM which will be studied in Unit 3, but students may be able to deduce an intuitive idea of what is going on here.)

What happens in a similar situation but where the acceleration depends on '$+x$' rather than '$-x$'? For example, suppose acceleration is $(-3 + x)$ measured in the positive $x$ direction and that at $t = 0$, $x = 1$ and $v = 2$.

By similar methods to above, show that: $v^2 = x^2 - 6x + 9$, which leads to: $v = -x + 3$ (discarding the second solution since $v = 2$ when $x = 1$).

We can see that if $x$ reaches the value '3' both the acceleration and velocity become zero and the particle is at rest. But does it ever reach that value? As $x$ approaches 3 both the velocity and acceleration get smaller and smaller. To investigate further write:

$\frac{dx}{dt} = -x + 3$ which can be solved to give: $-\ln(3 - x) = t - \ln 2$

Rearranging gives: $x = 3 - 2e^{-t}$

A sketch of the function would be useful and the idea of a limiting value as $t$ becomes large could be discussed.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When faced with a variable acceleration question, some students still try to use the *suvat* equations.

When variable acceleration formulae are used, some common errors are: using the wrong form for the acceleration and not calculating the constant of integration. Some students seem not to realise that they can find $a$ using $v\frac{dv}{dx}$.

When answering 'show that' questions students should be reminded that all steps must be shown to earn the marks. For example, when calculating the integration constant it is not sufficient to simply write down $c = 0$; the substitution needs to be shown since the answer is given.

The difference between displacement and distance is frequently either ignored or goes unrecognised. Students should be aware that in some situations a particle may re-trace its previous path and that they must split the integral to calculate distance rather than displacement.

# A level Further Mathematics: Mechanics 3

## UNIT 2: Elastic strings and springs and elastic energy

Return to Overview

### SPECIFICATION REFERENCES

**2.1** Elastic strings and springs. Hooke's law

**2.2** Energy stored in an elastic string or spring

### PRIOR KNOWLEDGE

- Intuitive idea of elasticity in strings and springs

IAS Mathematics – Pure content

- Basic integration (Unit 5 of the P1 SoW and Unit 8 of the P2 SoW)

IAL Mathematics – Mechanics content

- Types of forces, Newton's laws, resolving forces, friction, equilibrium and dynamics (Units 4 and 5 of the M1 SoW)
- KE, GPE and the work–energy principle (Unit 3 of the M2 SoW)

### KEYWORDS

String, spring, light, elasticity, modulus of elasticity ($\lambda$), extension, natural length, elastic potential energy (EPE), kinetic energy (KE), gravitational potential energy (GPE), joules (J), conservation of energy, equilibrium, work–energy principle, Newton's 2nd law of motion, work done, joined or parallel strings/springs, friction, coefficient of friction, inclined plane.

# A level Further Mathematics: Mechanics 3

| **2a. Hooke's law and definition of modulus of elasticity**<br>**Derivation of elastic potential energy formula (2.1) (2.2)** | **Teaching time**<br>6 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- be able to investigate the ability of strings to stretch and springs to stretch and compress;
- be able to define the modulus of elasticity ($\lambda$), natural length ($a$) and extension ($x$);
- be able to use the above definitions to work out the tension in a stretched string or a stretched/compressed spring i.e. use Hooke's Law, $T = \frac{\lambda x}{a}$ ;
- be able to derive the elastic potential energy (EPE) from Hooke's Law by applying the work done in stretching a string/spring i.e. EPE = $\frac{\lambda x^2}{2a}$.

## TEACHING POINTS

This topic could be introduced by an experiment in which a string or spring has varying loads put on it. The load and amount of stretch (or extension) of the string can be plotted and should give a straight line graph (up to a limit). This shows that the extension is directly proportional to the applied load for a given string/spring.

What if we change to a different size or type of string/spring? The gradient should be different.

This investigation was performed by Robert Hooke (1635–1703). The unstretched length of a spring or string is defined as the natural length, $a$, the extension (or compression of a spring) is denoted $x$, and the force needed to *double* the unstretched length of a string (or halve the uncompressed length of a spring) is called the modulus of elasticity, $\lambda$.

[Take care that this is not confused with the spring constant $k$ used in physics.]

If a string or spring is 'stronger', then $\lambda$ is greater, as a greater force is needed to double the length.

Hooke's experiments found that the extension was directly proportional to the applied load (i.e. tension) and he proposed the following relationship connecting the modulus of elasticity, the extension and the natural length with the tension in a stretched string or spring:

$$T = \frac{\lambda x}{a}$$

(In SI units both $T$ and $\lambda$ are measured in newtons.)

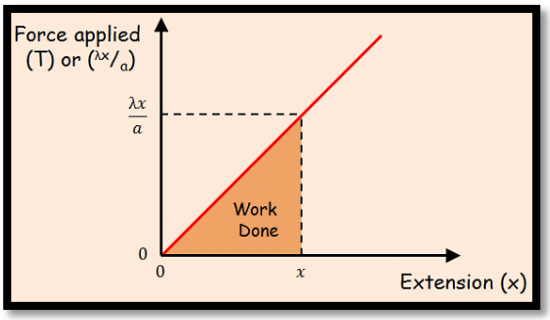
This became known as Hooke's Law.

Note that if a string (or spring) is stretched too much it will deform and not return to its maximum length when released i.e. it has exceeded its 'elastic limit' and Hooke's Law will not apply. However, questions will refer to a 'perfectly elastic' string so that it can be assumed Hooke's Law is applicable.

Remember that Hooke's Law also applies to springs which are compressed with the thrust being proportional to the compression.

# A level Further Mathematics: Mechanics 3

Work done in stretching a string




We know that, for a constant force:
work done = force × distance

But here force varies as we stretch the string. In other words, the tension increases as the extension increases, so tension $T$ is not constant.

For a variable force, work done = $\int F \, ds$

So in this case we write:

Work done (in stretching the string from its natural length through an extension $x$)

$$= \int F \, ds = \int T \, dx = \int_0^x \frac{\lambda x}{a} dx = \frac{\lambda x^2}{2a}$$

Note that this implies the work done is equivalent to area under the graph (as we would expect).

Therefore the work done in stretching the string from zero extension (natural length) to an extension of $x$ is $\frac{\lambda x^2}{2a}$ and it is called the elastic potential energy (EPE). If SI units are used for force and length then the units of EPE are joules.

We can think of the EPE as energy stored in a string or spring, with the potential for it to be transformed into other forms of energy during the subsequent motion.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It is important to point out that the EPE is the work done to reach an extension/compression of $x$ but that if it is necessary to know the work done to reach an extension/compression of $x_2$ from one of $x_1$ then:

Work done $= \frac{\lambda(x_2^2 - x_1^2)}{2l}$

More general problem solving will be considered in the next sub-unit.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

These questions are often well answered with the most common errors due to arithmetic slips and the misreading (or misunderstanding) of the question; for example, using $40g$ after being told the weight was 40 N.

## NOTES

The next section of this topic will look at how we can use Hooke's law to solve problems involving equilibrium and the use of either the work-energy principle or the conservation of mechanical energy to solve problems involving the motion of a particle attached to an elastic string or a spring.

# A level Further Mathematics: Mechanics 3

| **2b. Problem solving: equilibrium and using the work–energy principle (2.2)** | **Teaching time** 5 hours |
|---|---|

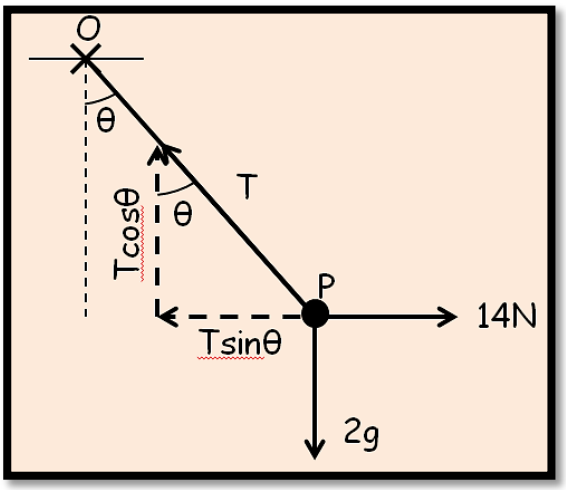
## OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate the tension in a string or spring when a system is held in equilibrium;
- be able to include EPE when using the work–energy principle;
- know the conditions for conservation of mechanical energy;
- be able to solve string/spring problems involving work and energy (i.e. KE, GPE and EPE).

## TEACHING POINTS

For systems in equilibrium (like the example below), you will need to resolve forces (making the resultant force = 0 N) and apply Hooke's law.



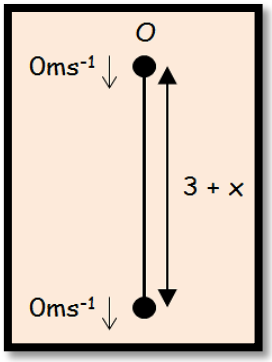

A particle could be attached to the ends of two different strings and held in equilibrium as in the example below:

Fixed points $A$ and $B$ are on a horizontal ceiling, where $AB = 4a$. A light elastic string has natural length $3a$ and modulus of elasticity $\lambda$. One end of the string is attached to $A$ and the other end is attached to $B$. A particle $P$ of mass $m$ is attached to the midpoint of the string. The particle hangs freely in equilibrium at the point $C$, where $C$ is at a distance $\frac{3}{2}a$ vertically below the ceiling. Show that $\lambda = \frac{5mg}{4}$

Note that if a particle is *attached* to a point on a string then effectively it becomes a two string problem. The modulus of elasticity is unchanged for each part, and the sum of the natural lengths will be the natural length of the whole original string. In the above example the natural lengths of the 'half strings' will be $1.5a$ each. Vertical resolution and Hooke's Law lead to the required result.

# A level Further Mathematics: Mechanics 3

When a system is not in equilibrium and the particle attached to an elastic string/spring is moving, the '$F = ma$' equation is complicated by the fact that the acceleration is not constant. In this specification the approach used to solve these problems is to consider the work–energy principle or conservation of mechanical energy. This is illustrated in the question below.




A particle of mass 0.6 kg is attached to one end of a light elastic string, of natural length 3 m and modulus of elasticity 9.8 N. The other end of the elastic string is attached to the fixed point O.

The particle is released from the point O.

Find the greatest distance reached by the particle below O.

The greatest distance the particle reaches below O occurs when the particle is instantaneously at rest.

In this case, there are no external forces (other than gravity) and the KE at the start and end is zero (greatest distance below O), so energy is conserved and we can equate the sum of GPE and EPE at the start with that at the end. In other words the loss in GPE = gain in EPE

It is of interest to note that the maximum speed of the particle occurs as it passes through its equilibrium position.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Further problem solving may include motion on a rough plane and consequently the work done against friction in addition to changes in energy. The first example below involves a horizontal plane and the second example involves an inclined plane. Both examples require the derivation of an inequality; it is important that students realise that $F = \mu R$ can only be assumed if the particle is moving or in limiting equilibrium; otherwise the inequality $F \leq \mu R$ must be applied.

A particle $P$ of mass $m$ is attached to one end of a light elastic string of natural length $l$ and modulus of elasticity $3mg$. The other end of the string is attached to a fixed point $O$ on a rough horizontal table. The particle lies at rest at the point $A$ on the table, where $OA = \frac{7}{6}l$.

The coefficient of friction between $P$ and the table is $\mu$.

**(a)** Show that $\mu \geq \frac{1}{2}$

The particle is now moved along the table to the point $B$, where $OB = \frac{3}{2}l$, and released from rest.

**(b)** Given that $\mu = \frac{1}{2}$, find the speed of $P$ at the instant when the string becomes slack.

* See Examiners Report below

One end $A$ of a light elastic string $AB$, of modulus of elasticity $mg$ and natural length $a$, is fixed to a point on a rough plane inclined at an angle $\theta$ to the horizontal. The other end $B$ of the string is attached to a particle of mass $m$ which is held at rest on the plane. The string $AB$ lies along a line of greatest slope of the plane, with $B$ lower than $A$ and $AB = a$. The coefficient of friction between the particle and the plane is $\mu$, where $\mu < \tan\theta$. The particle is released from rest.

**(a)** Show that when the particle comes to rest it has moved a distance $2a(\sin\theta - \mu\cos\theta)$ down the plane.

**(b)** Given that there is no further motion, show that $\mu \geq \frac{1}{3}\tan\theta$

* See Examiners Report below

In part **(b)** of the above example it is important to realise that the particle has already slid down the plane and come to rest so any further motion will be up the plane. It is therefore necessary to show that the tension in the string at the point where the particle stopped is less than or equal to the sum of the component of the weight down the plane and the maximum possible friction force.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The following quote relates to the first example above:

The first part highlighted the difficulty many candidates have in setting out a plausible proof, especially when an inequality is involved. All too often the tension in the string was correctly equated to the frictional force, which was then incorrectly stated to be equal to (rather than less than or equal to) $\mu R$. This showed a poor understanding of friction forces.

Once $\mu = \frac{1}{2}$ had been obtained it was common to see the required inequality appear. Sometimes an attempt to explain the inequality was given, but only in rare cases would this be satisfactory.

Most attempted the second part using the change in energy, successfully equating elastic potential energy, kinetic energy and work done against friction although there were some sign errors in the equations. However, many omitted the work done against friction from their calculation, in spite of the attention drawn to friction by completing the first part of the question.

The following quote relates to the second example above:

Many students did not recognise part **(a)** as a work–energy problem and tried to find an acceleration which they could use, ignoring the fact that the acceleration was in fact variable; others mixed forces and energy terms in their equation. This is a 'show that' question and so students must show every step of their working.

# A level Further Mathematics: Mechanics 3

## UNIT 3: Further dynamics

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### SPECIFICATION REFERENCES

3.1 Newton's laws of motion, for a particle moving in one dimension, when the applied force is variable

3.2 Simple harmonic motion. Oscillations of a particle attached to the end of elastic string(s) or spring(s)

### PRIOR KNOWLEDGE

- Hooke's Law for elastic strings/springs

IAL Mathematics – Pure content

- Trigonometric functions in radians (Unit 6 of the P2 SoW)
- Differentiation (Unit 4 of the P1 SoW and Unit 4 of the P3 SoW)
- Integration (Unit 5 of the P1 SoW, Unit 5 of the P3 SoW and Unit 6 of the P4 SoW)
- Differential equations (Unit 6 of the P4 SoW)

IAL Mathematics – Mechanics content

- Resolving forces, dynamics (Unit 4 of the M1 SoW)
- Power (Unit 3 of the M2 SoW)

### KEYWORDS

Force, acceleration, variable, mass, velocity, displacement, speed, distance, integration, differential equation, gravitational force, non-gravitational force, power, proportional, inversely proportional, limiting velocity, gravitation, radius, force of attraction, inverse square law, escape velocity, simple harmonic motion, oscillation, amplitude, period, reference circle, vertical, horizontal, elastic string, spring.

### NOTES

This unit is concerned with the motion of a particle moving in straight line with a variable applied force. Newton's second law, $F = ma$, is used to set up a differential equation. The first sub-unit builds on Unit 1 (Further kinematics) and it requires the solution of first order differential equations similar to those previously considered. Again the specification states the solution of the resulting equations 'will be consistent with the level of calculus required in units P1, P2, P3 and P4'.

Gravitation is investigated as a particular example of a variable force.

Simple harmonic motion is defined and used to describe oscillations. An understanding of elastic strings/springs is required (Unit 2).

# A level Further Mathematics: Mechanics 3

| **3a. Particle moving in straight line with variable applied force. Using *F* = *ma* to set up differential equations and solving (3.1)** | **Teaching time** 6 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- understand that, when a variable force (function of $t$ or $x$) acts on a particle moving in a straight line, acceleration may be written as $\frac{dv}{dt}$ or $v\frac{dv}{dx}$ or $\frac{d^2x}{dt^2}$;
- know how to use $F = ma$ to set up a first order differential equation in $v$ and $t$, or in $v$ and $x$;
- be able to solve the resulting differential equation using calculus;
- know how to apply boundary/initial conditions to find arbitrary constants.

## TEACHING POINTS

We consider the motion of a particle in a straight line under the action of a variable force (function of $x$ or $t$). By applying Newton's second law $F = ma$ and choosing the appropriate expression for the acceleration, we can set up a first order differential equation either in $v$ and $t$ or in $v$ and $x$.

The nature of the resulting equation and application of boundary conditions are similar to those encountered previously in Unit 1. A typical example is given below:

> A particle $P$ of mass $\frac{1}{3}$ kg moves along the positive $x$-axis under the action of a single force.
>
> The force is directed towards the origin $O$ and has magnitude $\frac{k}{(x+1)^2}$ N, where $OP = x$ metres and $k$ is a constant.
>
> Initially $P$ is moving away from $O$. At $x = 1$ the speed of $P$ is 4 m s$^{-1}$, and at $x = 8$ the speed of $P$ is $\sqrt{2}$ m s$^{-1}$.
>
> Find the value of $k$.

Since the force is a function of $x$, write the equation of motion (noting that the force is directed towards origin) as:

$$\frac{1}{3}v\frac{dv}{dx} = -\frac{k}{(x+1)^2}$$

Separating the variables and integrating gives:

$$\frac{1}{2}v^2 = \frac{3k}{x+1} + c$$

By substituting the two sets of boundary conditions, two simultaneous equations in the constants are obtained which can be solved.

*(**Answer**: $k = 6$)*

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Applying the impulse momentum equation $Ft = mv - mu$ and choosing an appropriate expression for the force we can set up another differential equation in $v$ and $t$. For a variable force $F(t)$, Newton's

second law, $F(t) = ma$, can be written as $F(t) = m\frac{\mathrm{d}v}{\mathrm{d}t}$. If the particle has constant mass and a velocity $U$ at time $t_1$ and a velocity $V$ at time $t_2$, then separating variables, integrating and applying the limits gives:

$$\int_{t_1}^{t_2} F(t)dt = m\int_U^V 1dv \;\Rightarrow\; \int_{t_1}^{t_2} F(t)dt = m(V-U) = mV - mU$$

$mV - mU$ is the change in momentum of the particle.

Since Impulse = change in momentum, $\int_{t_1}^{t_2} F(t)\mathrm{d}t$ is the impulse of the variable force on the particle.

In Mechanics 2 we saw that the work done by a force which accelerates a particle horizontally is equal to the change in kinetic energy: $Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$. For a variable force $G(x)$, Newton's second law, $G(x) = ma$, can be written as $G(x) = mv\frac{\mathrm{d}v}{\mathrm{d}x}$. If the particle increases its speed from $U$ to $V$ while moving from a point $x = x_1$ to a point $x = x_2$, then separating variables, integrating and applying the limits gives:

$$\int_{x_1}^{x_2} G(x)dx = m\int_U^V v\,dv = \frac{1}{2}mV^2 - \frac{1}{2}mU^2$$

So $\int_{x_1}^{x_2} G(x)dx$ is the work done by the variable force $G(x)$ which moves the particle from the point $x = x_1$ to a point $x = x_2$.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors involve incorrect signs in the initial differential equation, problems in substituting the limits and simplifying the answer, not being able to progress with the integration and simply stating without justification that $c = 0$.

## NOTES

In this unit air resistance is treated as negligible or modelled as a constant force. However, it might be useful to point out that, in reality, the magnitude of resistance depends on the speed of the particle. As a general rule it is taken to be proportional to the speed when this is relatively low, and proportional to the square of the speed when it is high. As a result, acceleration is often of the form $a + bv$ or $a + bv^2$ (where $a$ and $b$ are constants).

Newton's law of gravitation is an example of a variable force (varies inversely as the square of distance). It is studied in detail in the next section. We will mainly be concerned with the motion of a particle moving either towards or away from the Earth.

# A level Further Mathematics: Mechanics 3

| **3b. Newton's law of gravitation (3.1)** | **Teaching time**<br>4 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- know and understand Newton's law of gravitation;
- be able to use Newton's law of gravitation to set up a differential equation;
- be able to solve problems involving the motion of a particle moving towards or away from the centre of the Earth.

## TEACHING POINTS

Although Kepler, an astronomer in the early seventeenth century, formulated laws describing planetary motion, it took Newton to realise that these were, in fact, natural consequences of a far more general law. He put forward the idea that all masses were attracted to each other by an 'inverse square law', that is:

> The force of attraction between two bodies of mass $M_1$ and $M_2$ is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Mathematically this is written as:

$$\boldsymbol{F = G\frac{M_1M_2}{r^2}}$$

$G$ is called the gravitational constant. Its approximate value was determined many years after Newton by Henry Cavendish in 1798. This approximation has been improved on over more recent years and, to 3 significant figures, the value of $G$ is taken to be $6.67 \times 10^{-11}$.

Note that $G$ is not a dimensionless constant. Using the formula with SI units gives the units for $G$ as $\text{kg}^{-1}\,\text{m}^3\,\text{s}^{-2}$.

Note also that $r$ is the distance between the centres of mass of the two bodies. Unless at least one of the masses is very large this force has negligible magnitude. For example, two masses of 1 kg at a distance 1 m apart would have a force of attraction of only $6.67 \times 10^{-11}$ N.

However, if one mass is the Earth, this force is far from negligible and it accounts for the 'acceleration due to gravity, $g$' that is used throughout mechanics.

So what is the connection between $G$ and $g$?

Consider a particle of mass $m$. At the surface of the Earth, the force acting towards the centre of the Earth is:

$$F = mg$$

However, using Newton's law of gravitation, this force is:

$$F = \frac{GmM}{R^2}$$

where $M$ is the mass of the Earth, $R$ is the radius of the Earth (distance from centre of the Earth when the particle is on the surface and assuming that the Earth is a sphere).

Equating these two formulae gives:

$$g = \frac{GM}{R^2}$$

# A level Further Mathematics: Mechanics 3

Although it is possible to use numerical values for the constants, often problems are solved in terms of these constants. *GM* (or even *GmM*) may be specified as a single constant (see question below).

In general, the equation of motion of a particle moving in a straight line either towards the centre of the Earth or away from it can be written down in the form of a first order differential equation in terms of $v$ and $x$. This can be solved by separating the variables and applying given boundary conditions. This general strategy can best be illustrated by a typical question:

> A spacecraft *S* of mass *m* moves in a straight line towards the centre of the Earth.
> The Earth is modelled as a sphere of radius *R* and *S* is modelled as a particle.
> When *S* is at a distance $x$ ($x \geq R$) from the centre of the Earth, the force exerted by the Earth on *S* is directed towards the centre of the Earth. The force has magnitude $\frac{K}{x^2}$ where *K* is a constant.
>
> (a) Show that $K = mgR^2$.
>
> When *S* is at a distance $3R$ from the centre of the Earth, the speed of *S* is *V*.
> Assuming that air resistance can be ignored,
>
> (b) find, in terms of *g*, *R* and *V*, the speed of *S* as it hits the surface of the Earth.

Part (b) requires writing the equation of motion as a differential equation:

$$\frac{mgR^2}{x^2} = -mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

(Note the *negative* sign because the force is towards centre of Earth which is opposite to the direction of $x$ increasing)

$g\int \frac{R^2}{x^2}\mathrm{d}x = -\int v\,\mathrm{d}v$ (separation of variables)

$-g\frac{R^2}{x} = -\frac{1}{2}v^2 \ \ (+c)$ (integration)

$x = 3R, v = V \Rightarrow -g\frac{R^2}{3R} = -\frac{1}{2}V^2 + c$ (applying boundary condition)

$\frac{1}{2}v^2 = -\frac{Rg}{3} + \frac{1}{2}V^2 + g\frac{R^2}{R}$ (substituting $x = R$ since we are looking for $v$ when particle is at surface)

$v = \sqrt{V^2 + \frac{4Rg}{3}}$ (rearranging gives the required value of $v$ in terms of the specified constants)

Note that the displacement, $x$, is generally measured from the centre of the Earth. However, in some questions, it may be measured from the surface of the Earth in which case the force of attraction is: $F = \frac{GmM}{(R+x)^2}$ and care must be taken when applying boundary conditions.

# A level Further Mathematics: Mechanics 3

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

An interesting idea for discussion is the concept of 'escape velocity'.

How fast would you need to fire an object to ensure that it never returns to Earth?

Consider the differential equation and solution given above:

$$-g\frac{R^2}{x} = -\frac{1}{2}v^2 \quad (+c)$$

Now suppose the speed of projection is $u$ (when $x = R$):

Substituting and finding $c$ gives:

$$-\frac{gR^2}{x} = -\frac{1}{2}(v^2 - u^2) - gR$$

Now if $v = 0$ the particle will have reached its greatest height.

If $u^2 < 2gR$ the equation can be solved for $x$. However, if $u^2 \geq 2gR$ the particle will never reach a greatest height and it will continue to move away from the Earth. Note that in this model air resistance and any other forces acting on the particle are ignored. If we model the Earth as a sphere of radius $6.37 \times 10^6$ m, the escape velocity is $1.12 \times 10^4$ m s$^{-1}$.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students need to be careful in 'show that' questions; marks were lost by those who did not make it clear that they were considering the particle to be on the surface of the Earth. Remind students that in these sorts of questions all steps need to be explained and justified.

Students should be reminded to read carefully the information given as marks can be easily lost by using incorrect information, for example by taking $x$ as the distance from the centre of the earth rather than $R + x$ when this is given in the question.

## NOTES

For all problem solving the Earth will be assumed to a fixed sphere and air resistance will be ignored.

# A level Further Mathematics: Mechanics 3

| **3c. Simple harmonic motion (3.2) (3.3)** | **Teaching time**<br>5 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

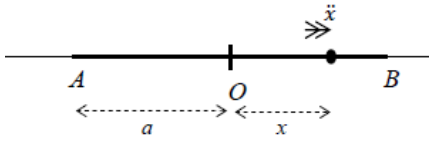
- be familiar with the defining equation of SHM and its solutions;
- know and understand the formulae for amplitude, period, frequency, velocity;
- be able to apply SHM theory to problem solving involving oscillations (elastic strings/springs).

## TEACHING POINTS

Within this unit we have been considering the motion of a particle under the action of a variable force. In this sub-unit we are looking at one particular example of this where the force is proportional to the distance of the particle from a fixed central point and is always directed towards this point. This is called simple harmonic motion (SHM) and it underlies many naturally occurring phenomena.

Think intuitively about what this means. A particle accelerates under the action of a force towards a fixed point. As the particle gets near to the point the force (and therefore the acceleration) becomes smaller; actually at the point there is no force acting but the particle will have a velocity so the particle will continue past the point. However, now the particle is moving away from the point so the force will cause it to slow down and eventually come to instantaneous rest. It will then accelerate back towards the point and so the motion will continue backwards and forwards in oscillations. The fixed point is the centre of oscillation.

How can this motion be defined mathematically?




If $x$ is the displacement from a fixed point, $O$:

$$\ddot{x} = -\omega^2 x$$

Acceleration, $\frac{d^2x}{dt^2}$ (which is equivalent to $\frac{dv}{dt}$), is written using the established convention that two dots represent the second derivative with respect to time (one dot is first derivative). Note that we use $\omega^2$ for the constant and not just $\omega$ to ensure that it is positive. The minus sign reflects the fact that the acceleration is directed *towards* the fixed point. (Is there a connection between this $\omega$ and the $\omega$ describing angular speed in circular motion? The answer is yes and we will consider this connection later.)

There are two alternative approaches for solving this differential equation.

If general solutions of second order linear differential equations have already been covered then students will have seen this type of equation before. They should be able to write down the general solution in terms of sines and cosines.

However, it can also be solved by similar methods to those previously employed in this unit. Write the differential equation in terms of $v$ and $x$:

$$v\frac{dv}{dx} = -\omega^2 x$$

Separate the variables and integrate to give:

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$$

When the speed is zero, $x$ has its maximum or minimum value. So when $v = 0$, $x = +a$ or $-a$ (where $a$ is the greatest distance from the fixed point, $O$).

Hence we have a formula for the speed at any point of the oscillation:

$$\boldsymbol{v^2 = \omega^2(a^2 - x^2)}$$

[Note that the maximum $v$ ($= a\omega$) occurs when $x = 0$ (centre of oscillation) as would be expected.

It is important to remember the negative root when square rooting. Also note that for any value of $x < a$ there are two values of $v$ (one positive and one negative) and we get the same values of $v$ when we substitute $-x$ rather than $x$. This reflects the symmetry of the oscillation.]

We can then square root, separate variables and integrate (using trig substitution in $(a^2 - x^2)^{-\frac{1}{2}}$ or quoting the result from the formula book).

We are interested in two solutions in particular:

the first where we measure time from the fixed point (centre of oscillation) i.e. $t = 0$ when $x = 0$,

and the second where we measure time from an end point of the oscillation i.e. $t = 0$ when $x = a$.

These solutions are:

$$\boldsymbol{x = a \sin \omega t}$$
$$\boldsymbol{x = a \cos \omega t}$$

By differentiating with respect to $t$ we can obtain formulae for the velocity at time $t$.

We can see that the particle will oscillate between $x = a$ and $x = -a$ where $a$ denotes the amplitude.

The time for one complete oscillation is called the period. Looking at the solution for $x$ we can see that

$a \sin \omega t = a \sin(\omega t + 2\pi) = a \sin(\omega[t + \frac{2\pi}{\omega}])$.

This means that the period of the motion is given by:

$$\boldsymbol{T = \frac{2\pi}{\omega}}$$

Students should know and be able to quote the formulae in the boxes. Set a number of questions to test their use and to build up familiarity, for example:

A particle $P$ moves with simple harmonic motion and comes to rest at two points $A$ and $B$ which are 0.24 m apart on a horizontal line. The time for $P$ to travel from $A$ to $B$ is 1.5 s. The midpoint of $AB$ is $O$. At time $t = 0$, $P$ is moving through $O$, towards $A$, with speed $u$ m s$^{-1}$.

(a) Find the value of $u$.

(b) Find the distance of $P$ from $B$ when $t = 2$.

(c) Find the speed of $P$ when $t = 2$.

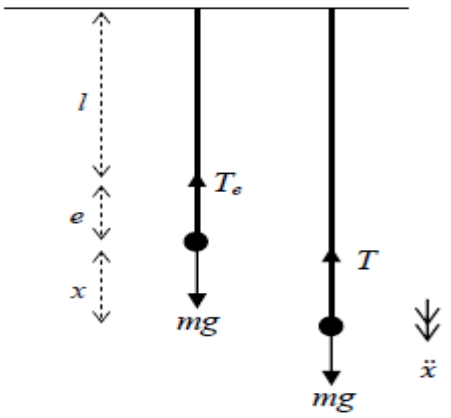
(***Answers***: *$u = 0.251$; distance $= 0.016$ m; speed $= 0.13$ m s$^{-1}$*)

The easiest practical demonstration of oscillations is using a weight on the end of a spring held vertically. In reality, the weight will gradually slow down and come to rest because of air resistance, but we shall treat air resistance as negligible in our modelling. A knowledge of Hooke's law is needed for analysis of the motion.

# A level Further Mathematics: Mechanics 3

A diagram is important. The mass will oscillate about its position of equilibrium. If the mass is initially pulled down a distance $a$ below the position of equilibrium and released from rest, then the amplitude of the ensuing oscillations will be $a$.

Note that if we have an elastic string rather than spring then $a$ must be smaller than $e$ (the equilibrium extension) for complete SHM – discus what would happen if $a >$ e.




We can show the particle moves with SHM by deriving the defining equation i.e.

In equilibrium, $T = mg$

By Hooke's law, $T = \frac{\lambda e}{l}$ ($e$ = equilibrium extension, $\lambda$ = modulus of elasticity, $l$ = natural length)

When extension is $(x + e)$, the equation of motion in the direction of $x$ increasing is:

$$mg - \frac{\lambda(e+x)}{l} = m\ddot{x}$$

Substituting $mg = \frac{\lambda e}{l}$ (from above):

$\ddot{x} = -\frac{\lambda l}{m} x$ which denotes SHM about $x = 0$ i.e. the equilibrium point.

Another naturally occurring example which is modelled as SHM is the height of tidal water between low and high tides. The centre of the motion will be the average of the two heights. Care must be taken with units; often time is measured in hours and related to clock times. An example of a typical question is given below:

A vertical ladder is fixed to a wall in a harbour. On a particular day the minimum depth of water in the harbour occurs at 09 00 hours. The next time the water is at its minimum depth is 21 15 hours on the same day.

The bottom step of the ladder is 1 m above the lowest level of the water and 9 m below the highest level of the water.

The rise and fall of the water level can be modelled as simple harmonic motion and the thickness of the step can be assumed to be negligible.

Find:

(a) the speed, in metres per hour, at which the water level is moving when it reaches the bottom step of the ladder

(b) the length of time, on this day, between the water reaching the bottom step of the ladder and the ladder being totally out of the water once more.

*(**Answers**: (a) 1.54 metres per hour (b) 9.7 hours or 9 hours 44 mins)*

# A level Further Mathematics: Mechanics 3

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be given the opportunity to solve a wide variety of problems including the use of two strings/springs with horizontal or vertical oscillations. The general strategies are similar:

- find the position of equilibrium
- measure the displacement ($x$) from this point and derive the SHM equation
- use standard formulae for $x$, $v$, $T$ as required.

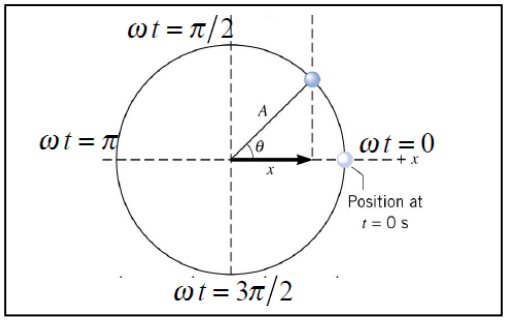
An example is given below:

> $A$ and $B$ are two points on a smooth horizontal floor, where $AB = 5$ m.
>
> A particle $P$ has mass 0.5 kg. One end of a light elastic spring, of natural length 2 m and modulus of elasticity 16 N, is attached to $P$ and the other end is attached to $A$. The ends of another light elastic spring, of natural length 1 m and modulus of elasticity 12 N, are attached to $P$ and $B$
>
> (a) Find the extensions in the two springs when the particle is at rest in equilibrium.
>
> Initially $P$ is at rest in equilibrium. It is then set in motion and starts to move towards $B$. In the subsequent motion $P$ does not reach $A$ or $B$.
>
> (b) Show that $P$ oscillates with simple harmonic motion about the equilibrium position.
>
> (c) Given that the initial speed of $P$ is $\sqrt{10}$ m s$^{-1}$, find the proportion of time in each complete oscillation for which $P$ stays within 0.25 m of the equilibrium position.

*(**Answers**: 1.2 m and 0.8; $\ddot{x} = -40x$ which is SHM; proportion =1/3)*

It is of interest to investigate an alternative way of analysing SHM. Previously it was indicated that there is a connection between the $\omega$ here and $\omega$ used as angular speed in circular motion.

If the path of a particle travelling in a circle with constant angular speed $\omega$ is projected onto the diameter, the projection of the particle will move with SHM along the diameter. This is often referred to as the reference circle or the auxiliary circle. The amplitude of oscillation will be the radius of the circle.




As the particle moves along the $x$-axis, using trigonometry it can be seen that

$x = A\cos\omega t$ ($t = 0$, $x = A$, radius of circle)

This representation is particularly useful for finding the time to move through a specified distance within an oscillation. We use trigonometry to find the required angle (in radians), and then divide by $\omega$ to find the time.

Look again at the question above. By symmetry it is only necessary to consider half a complete oscillation.

Maximum speed = $a\omega$ which gives $a = \frac{\sqrt{10}}{\sqrt{40}} = 0.5$

Angle at centre = $\pi - 2\arccos\left(\frac{0.25}{0.5}\right)$

$= \frac{\pi}{3}$ and so proportion is $\frac{1}{3}$

# A level Further Mathematics: Mechanics 3

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Using $a$ for the acceleration is not advisable as $a$ is non-directional; in this question it could also be confused with the length, $a$. It is better to use $\ddot{x}$ from the start, in the direction of increasing $x$. Also, if there is a requirement to prove SHM, there must be a conclusion stating that the motion is SHM, thereby indicating to the examiner that the proof is complete.

A common error is to use the wrong formula, for example in one question many candidates used $x = a\cos\omega t$ even though they had been informed that the particle was at the centre of the oscillation at the start of the motion.

## NOTES

The specification states that candidates will be expected to be familiar with standard formulae which may be quoted without proof.

# A level Further Mathematics: Mechanics 3

## UNIT 4: Motion in a circle

Return to Overview

### SPECIFICATION REFERENCES

**4.1** Angular speed

**4.2** Radial acceleration in circular motion. The forms $r\omega^2$ and $\frac{v^2}{r}$ are required

**4.3** Uniform motion of a particle moving in a horizontal circle

**4.4** Motion of a particle in a vertical circle

### PRIOR KNOWLEDGE

IAL Mathematics – Pure content

- Radians (Unit 2 of the P1 SoW)

IAL Mathematics – Mechanics content

- Modelling assumptions made throughout the M1 and M2 courses (e.g. particle, light, rod, inextensible, smooth)
- Resolving using Newton's Second Law: $F = ma$, friction (Unit 4 of M1 SoW)
- Projectiles (Unit 1 of M2 SoW)

### KEYWORDS

Circle, uniform, linear velocity, angular speed, angular acceleration, radians, revolution, radial, tangential, frequency, period, complete circle, central force, tension, normal reaction, string, rod, peg, inextensible, elastic, light, taut, slack, projectile, conical pendulum, hemispherical bowl, smooth, bead, wire, banking, friction, slipping, coefficient of friction.

### NOTES

This unit is concerned with circular motion, both horizontal where the speed is constant and vertical where the speed is variable. An understanding of potential and kinetic energy and the principle of conservation of mechanical energy in the context of motion in a vertical circle will be required.

# A level Further Mathematics: Mechanics 3

## 4a. Angular speed, central force, radial acceleration (4.1) (4.2)

**Teaching time**
3 hours

### OBJECTIVES

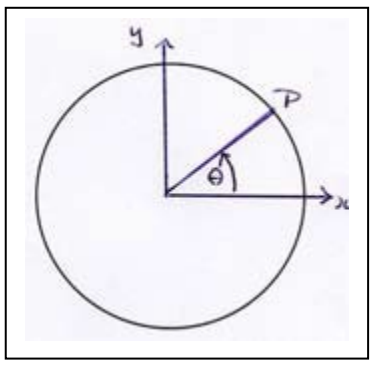
By the end of the sub-unit, students should:

- understand the difference between linear velocity and angular velocity;
- be able to write down and use the formula $v = r\omega$;
- know how to convert between units (for example) rad $s^{-1}$ and revs $min^{-1}$;
- understand that circular motion requires a central force;
- know that the acceleration is towards the centre of the circle and has magnitude $\frac{v^2}{r}$ or $r\omega^2$.

### TEACHING POINTS

Consider a particle moving in a circle. How could we describe how fast it is moving?

The speed can be measured in two ways: either a *linear* measure in, for example, metres per second, or alternatively an *angular* measure in, for example, revolutions per minute or radians per second.




If the particle $P$ moves in the circle shown then the angular speed is defined as $\frac{d\theta}{dt}$.

Practise conversion of units, in particular between rad $s^{-1}$ and rev $min^{-1}$.

Go on to consider the relationship between angular speed and linear speed. Imagine two children sitting next to each other on a roundabout with one being closer to the centre than the other. As it rotates the children remain next to each other and their angular speeds will be the same. However the child on the outside will travel further in one revolution than the one on the inside; hence their linear speed will be greater. The greater their distance from the centre the 'faster' they will be moving.

We can formalise this relationship by writing down the arc length formula: $s = r\theta$ (where $r$ is the radius).

Differentiating with respect to $t$ gives: $\frac{ds}{dt} = r\frac{d\theta}{dt}$ as $r$ is constant for a circle.

Denoting angular speed by $\omega$ we have the important relationship:

$$v = r\omega$$

Period of revolution is the term that describes how long it takes to go once round the circle.

If $\omega$ is measured in rad $s^{-1}$, it follows that:

$$T = \frac{2\pi}{\omega}$$

When a particle moves in a circle with a constant speed the direction of its linear velocity (which is tangential to the circle) is constantly changing; this means there is an acceleration (even though the speed is constant). This acceleration is always directed towards the centre of the circle (radial acceleration) and the magnitude is given by:

$$a = \frac{v^2}{r} = r\omega^2$$

A formal derivation can be found in most text books.

Using Newton's second law, the force required to produce this acceleration must have magnitude $\frac{mv^2}{r}$ and be directed towards the centre of the circle. This is often referred to as the centripetal force.

If you are going round on a fast fairground ride you feel as though you are being thrown outwards. This imagined force is sometimes referred to as a 'centrifugal force'. However, in reality there is no outwards force; it is just your tendency to want to continue moving in a straight line (Newton's first law of motion). The side of the car you are leaning on supplies the force that makes you move in a circle.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It may be worth showing a video clip of, for example, a hammer thrower to show how when the hammer is released (i.e. there is no longer a central force keeping it moving in a circle) it flies off tangentially from the circular path. This is because the linear velocity is tangential at any point.

Set some questions involving substituting values into the formulae stated above. For example, consider a clock hand 4 cm long. Find out how fast (linear speed) the tip of the hand is moving.

Students should become entirely familiar with the terminology, units and formulae before embarking on the next two sub-units which will involve extensive problem solving.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

*Common misconceptions are more evident in the next two sub-units where particular scenarios which lead to problem solving are considered.*

However, examiners continue to comment on errors in exam questions involving 'the supposedly straightforward conversion of revs/minute into rads/second.'

Students should also be aware that just using a calculator to generate a value for the angular acceleration results in no marks if an error is made. They should be advised to always quote any formulae they are using so that method marks may be awarded.

## NOTES

The theory in this section is applied to uniform motion in a horizontal circle and to vertical circular motion in the next two sub-units.

# A level Further Mathematics: Mechanics 3

| **4b. Uniform motion in a horizontal circle (4.3)** | **Teaching time**<br>6 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- understand that a particle moving in a horizontal circle with constant speed has an acceleration of magnitude $r\omega^2$ directed towards the centre of the circle;
- know that the resultant force acting on the particle has magnitude $mr\omega^2$ and is directed towards the centre of the circle;
- be able to model a variety of physical situations which give rise to uniform circular motion;
- understand that a banked surface is a plane that is at an angle to the horizontal;
- be able to consider how conditions affect the described circular motion, for example the greatest and least possible angular speeds.

## TEACHING POINTS

Having established the principles of circular motion in the last sub-unit, we now look at physical situations which give rise to uniform motion in a horizontal circle. In these examples the force inwards is often provided by the tension in a string or the reaction between the particle and the surface on which it is moving, or a combination of components of each.

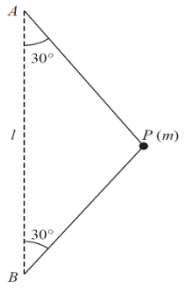
For each of the following examples it is a good idea to draw a diagram. Label the forces on the particle. Discuss where the centre of the circle is and which force, or combination of forces, provides the force towards the centre. Also note how the vertical forces balance.

a) Particle on the rough surface of a rotating disc – the friction force acts towards centre.

b) Conical pendulum (including elastic string) – a light string has one end attached to a fixed point and the other attached to a particle which is moving in a horizontal circle. The horizontal component of tension provides the central force. In the case of the elastic string, Hooke's law may be needed to find the tension in the string.

c) Smooth bead on a rotating wire – component of reaction acts towards centre of circular path.

d) Particle supported by two light strings – see diagram. Discuss why the tensions in this case cannot be equal and which one is greater. What is a condition for circular motion to be possible in this configuration with both strings taut? The fact that the tension in the lower string must be greater than zero leads to a minimum possible value for the angular speed (and consequently a maximum value for the period of revolution).

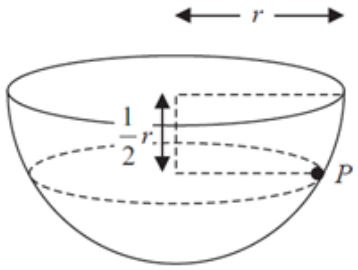



You could also discuss what happens to the tension in $AP$ as the angular speed increases leading to the possibility that the string breaks so that there is a minimum value for the period of revolution.

Note that if the particle is a smooth bead threaded on a string, then the tension must be the same in both parts of the string.

# A level Further Mathematics: Mechanics 3

e) Particle moving on a smooth fixed surface – for example a particle moving on the inside of a smooth fixed hemisphere as shown in the diagram.




Here it is important to note that:

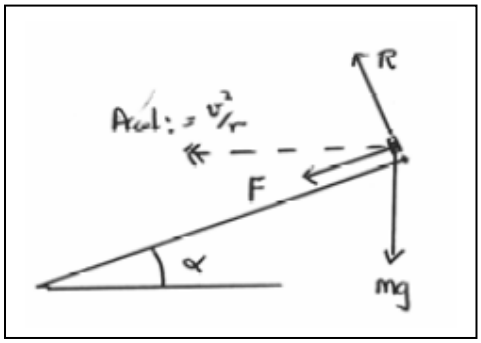
- the reaction is directed towards the centre of the hemisphere and **not** towards the centre of the circular path of the particle
- the radius of the circular path is **not** the same as the radius of the hemisphere (called $r$ here, so care must be taken when quoting standard formula '$r\omega^2$')

The radius of circle can be found by using Pythagoras' theorem and the angle of the reaction with horizontal is found by trigonometry.

f) Banking.

When a car rounds a bend whilst travelling on a horizontal road, there must be a central force enabling it to follow part of a circular path. The faster the car travels, the greater the required friction to prevent slipping. The magnitude of the force acting towards the centre of the circle can be increased if the surface of the road is inclined at an angle to the horizontal. This is described as a banked surface. On roads it helps to reduce wear on tyres as well as being safer. It is particularly evident on race tracks and it might be useful to show some photographs or video clips.

From the diagram it can be seen that the central force is now the resultant of the horizontal component of friction and the horizontal component of the normal reaction.




Since there is a maximum possible value of friction ($\mu R$) for no slipping up the slope, by resolving vertically and horizontally the maximum value of the speed, $v$, can be found.

It is worth noting that if the car is travelling slowly then the frictional force could be acting in the opposite direction (i.e. up the slope) to prevent the car from sliding down the slope.

An example of a typical question is:

A car of mass 800 kg is driven at constant speed $v$ m s$^{-1}$ round a bend in a race track. Around the bend, the track is banked at 20° to the horizontal and the path followed by the car can be modelled as a horizontal circle of radius 20 m. The car is modelled as a particle. The coefficient of friction between the car tyres and the track is 0.5.

Given that the tyres do not slip sideways on the track, find the maximum value of $v$.

*(**Answer**: 14.4 m s$^{-1}$)*

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## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Below is an example of a system which combines the principle of a conical pendulum and the motion of a particle on a smooth surface:

A cone of semi-vertical angle 60° is fixed with its axis vertical and vertex upwards. A particle of mass $m$ is attached to one end of a light inextensible string of length $l$. The other end of the string is attached to a fixed point vertically above the vertex of the cone. The particle moves in a horizontal circle on the smooth outer surface of the cone with constant angular speed $\omega$, with the string making a constant angle 60° with the horizontal, as shown in Figure 1.

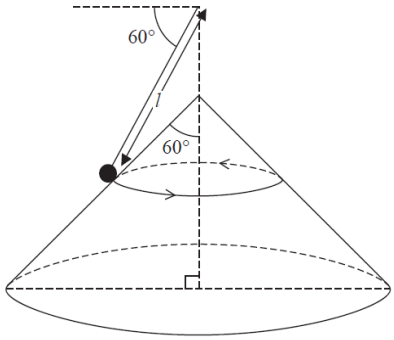


**Figure 1**

a) Find the tension in the string, in terms of $m$, $l$, $\omega$ and $g$.

The particle remains on the surface of the cone.

b) Show that the time for the particle to make one complete revolution is greater than $2\pi\sqrt{\frac{l\sqrt{3}}{2g}}$.

The general strategy is the same as usual:

Equate the total (horizontal) force acting towards the centre of the circular path to '$mr\omega^2$' and balance forces vertically.

However, the situation here is slightly more complicated and care must be taken in finding the correct trigonometric ratios to use when resolving. Note that $r$ is the radius of the circle in which the particle is moving.

For part (b) discuss the condition required for this motion to be possible. What might happen if the particle were to move really fast? It could leave the surface of the cone, and so the required condition is that the reaction remains positive (a hint here is given by 'the particle remains on the surface of the cone'). This will lead to an inequality for $\omega$ and hence the required condition for the value of the period of revolution.

Note that in many of these problems lengths and masses are given in symbolic form. If the answer is not given in the question, then it is worthwhile encouraging students to do a 'dimensions' check on any expressions they have obtained. Also, if square roots are involved, it is important that the square root sign clearly includes all the terms it is meant to.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Errors seen by examiners in horizontal circular motion questions include:

Students wrongly applying energy methods as if the question were a vertical circular motion question.

Students equating the horizontal component of the reaction to $mr\omega^2$ where $r$ was the given radius of the bowl and not the horizontal radius as required.

In two string problems, students not using the condition that the tension in the lower string has to be greater than or equal to zero and instead considering the difference in the two tensions.

## NOTES

In the next sub-unit ideas are extended to cover a particle moving in a vertical circle (or part of circle) where the speed is variable.

# A level Further Mathematics: Mechanics 3

| **4c. Motion in a vertical circle (4.4)** | **Teaching time**<br>6 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- understand that a particle moving in a vertical circle has variable speed;
- be able to find the radial and tangential components of acceleration for a particle moving in a vertical circle;
- understand and be able to use the principle of conservation of mechanical energy in this context;
- know how to model a variety of physical situations which give rise to vertical circular motion;
- understand the conditions required for complete circles in different scenarios;
- be able to use projectile equations to describe the motion of a particle after leaving a circular path and moving freely under gravity.

## TEACHING POINTS

When a particle moves in a circle with variable speed it can be shown that, as in uniform motion, the acceleration towards the centre is equal to $r\omega^2$ or $\frac{v^2}{r}$. However, now there will also be a tangential component of acceleration.

Tangential acceleration = $\frac{\mathrm{d}v}{\mathrm{d}t} = r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ (since $v = r\frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $r$ is constant).

The following are four possible scenarios in which a particle is constrained to move in a vertical circle and which demonstrate the principles involved:

a) Particle attached to a light inextensible string
b) Particle moving on the inside of a hollow smooth sphere/cylinder
c) Particle attached to a light rod
d) Particle moving on the outside of a smooth sphere/cylinder

Practical demonstrations could be given, for example a marble moving in a vertical circle in a hemispherical bowl. It would clearly be seen that the particle is moving fastest at the lowest point and that the speed is not constant.

The general strategy for analysing the motion is the same in all cases:

Resolution (using $F = ma$) towards the centre of the circle gives one equation.
Another equation is found from energy considerations.

The ideas of kinetic energy (KE = $\frac{1}{2}mv^2$) and gravitational potential energy (GPE = $mgh$) should be introduced. The force acting towards the centre of the circle at any instant is perpendicular to the direction of motion and therefore does no work; hence the change in KE can be equated to the change in GPE. If the tangential acceleration is required, this can be found by using $F = ma$ tangentially (or by differentiating the energy equation – this is sometimes a useful check).

If we are looking for the condition that the particle moves in *complete* circles then the situations in (a) and (b) are the same in terms of mechanics:

The tension/reaction must be greater than zero at the highest point of the circle.

Discuss where the maximum and minimum values of the tension/reaction occur.

The following question illustrates these points.

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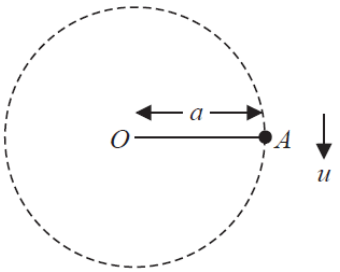
A particle of mass $3m$ is attached to one end of a light inextensible string of length $a$. The other end of the string is attached to a fixed point $O$. The particle is held at the point $A$, where $OA$ is horizontal and $OA = a$. The particle is projected vertically downwards from $A$ with speed $u$, as shown in the diagram. The particle moves in complete vertical circles.




(i) Show that $u^2 \geq 3ag$.

(ii) Given that the greatest tension in the string is three times the least tension in the string, show that $u^2 = 6ag$.

We use the condition that $T \geq 0$ at the highest point.

First consider changes in KE and GPE between $A$ and the highest point.

Then use $F = ma$ towards the centre at the highest point. Eliminate $v$ (speed at the highest point) and then set $T \geq 0$ to achieve the inequality.

Finally we use the fact that tension is least at the highest point and greatest at the lowest point.

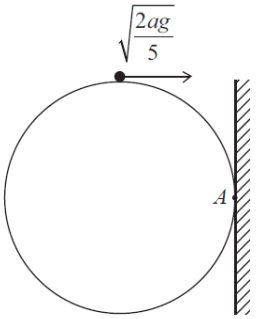
In case (c), the particle is attached to a rod which remains rigid so it cannot leave the circular path. Hence the condition for moving in *complete* circles is $v > 0$ at the highest point.

In (d) the particle will leave the surface of the sphere at the point when $R = 0$ where $R$ is the normal reaction on the particle due to the sphere.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

A fairly common problem involves considering the motion of a particle after it has left its circular path and moves freely under gravity as a projectile. This could arise when it leaves the surface of a sphere as in the example below. The condition $R = 0$ determines the starting point of this stage of the motion. Remember that the particle is moving tangentially at this point. To use projectile equations it is essential that vertical and horizontal components of velocities are used. Depending on the nature of the problem, it may be easier to equate changes in KE and GPE (but remember that at the highest point of a projectile's path the speed is not zero – there is a horizontal component).

A smooth sphere of radius $a$ is fixed with a point $A$ of its surface in contact with a fixed vertical wall. A particle is placed on the highest point of the sphere and is projected towards the wall and perpendicular to the wall with horizontal speed $\sqrt{\frac{2ag}{5}}$.




The particle leaves the surface of the sphere with speed $V$.

(a) Show that $V = \sqrt{\frac{4ag}{5}}$.

The particle strikes the wall at the point $X$.

(b) Find the distance $AX$.

The first part of the problem is solved by the usual strategy of resolving towards the centre of the circle and equating changes in KE and GPE. In the second part, when the particle is moving as a projectile, the time

can be found from working horizontally, and then this can be used in a 'vertical' equation to find the required distance.

*(**Answer**: $AX = \frac{11a}{32}$)*

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes are often made as students try to apply incorrect conditions or conditions that would apply in other scenarios.

Student also sometimes have difficulty identifying the direction of motion of the particle at the instant it leaves the surface of the sphere.

For one question, an examiner commented: 'The projectile problem could be solved by using vertical motion under gravity or by energy considerations. Many who opted for energy forgot that there was KE at the highest point; some of those who opted for vertical motion under gravity forgot to resolve the velocity.'

## NOTES

It is worth students attempting a large number of questions to become completely familiar with the full range of possible scenarios. To quote an examiner report *'Vertical circle questions regularly present problems for many candidates'*

# A level Further Mathematics: Mechanics 3

## UNIT 5: Statics of rigid bodies

Return to Overview

### SPECIFICATION REFERENCES

**5.1** Centre of mass of uniform rigid bodies and simple composite bodies

**5.2** Simple cases of equilibrium of rigid bodies

### PRIOR KNOWLEDGE

IAL level Mathematics – Pure content

- Integration, volume of revolution (Unit 6 of P4 SoW)

IAL Mathematics – Mechanics content

- Moments (Unit 6 of M1 SoW)
- Equilibrium and statics (Unit 5 of M1 SoW)
- Friction (Unit 4 of M1 SoW)
- Centre of mass of uniform plane figures (Unit 2 of M2 SoW)

### KEYWORDS

Mass, uniform, non-uniform, centre of mass, lamina, body, rigid, solid, area, density, algebraic integration, solid of revolution, volume of revolution, moment, sphere, hemisphere, right circular, cone, cylinder, shell, hollow, composite, resting, plane, inclined, equilibrium, limiting equilibrium, sliding, slipping, friction, coefficient of friction, toppling, freely suspended, angle, vertical, horizontal.

### NOTES

In this unit we derive, from first principles, formulae using integration for 2D and 3D shapes. We also consider composite bodies and their stability.

# A level Further Mathematics: Mechanics 3

| **5a. Centre of mass of uniform rod, lamina, 3D solid using integration (and symmetry); Deriving formulae in formula book (5.1)** | **Teaching time** 5 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the mass of a uniform rod, lamina or 3D solid using integration;
- be able to find the centre of mass of a uniform rod, lamina or 3D solid using integration (and symmetry);
- be able to derive the formulae for centres of mass of various shapes as specified in formula book.

## TEACHING POINTS

In earlier work the centre of mass of a lamina was found by symmetry and by combining standard results. In this sub-unit we extend the range of shapes and objects for which we can find the centre of mass by using integration. The same basic principle is used:

Sum of separate moments = moment of whole

To derive the integration formulae it is necessary to divide the shape into small 'elements' and sum the elements to find the total mass, and then sum the moments of the elements to enable the centre of mass to be calculated.

We make use of the general result:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

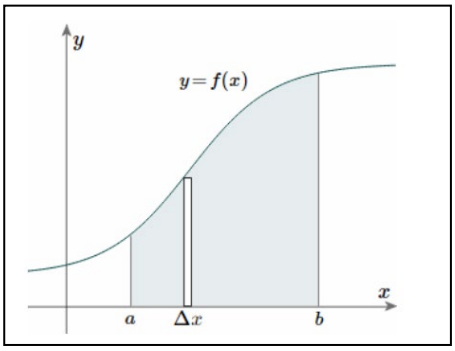
First consider a one dimensional rod.

Each element has mass $m_i = \rho\delta x$ where $\rho$ is mass per unit length and $\delta x$ is the length of the element.

Taking the limit as $\delta x \to 0$, the above sums become integrals and so:

$$\bar{x} = \frac{\int x\rho \mathrm{d}x}{\int \rho \, \mathrm{d}x}$$

Then apply the same technique to a lamina which is bounded by the curve $y = f(x)$, the $x$ axis and the lines $x = a$ and $x = b$.




Each element (strip) has area = $y\delta x$ and so $m_i = y\rho\delta x$.

Taking the limit as $\delta x \to 0$ leads to the formula:

$$\bar{x} = \frac{\int_a^b xy\rho \, \mathrm{d}x}{\int_a^b y\rho \, \mathrm{d}x}$$

Note that if the lamina is uniform ($\rho$ constant) then $\rho$ cancels out from the formula.

The corresponding $y$-coordinate can be found by noting that the centre of mass of our element is half way up the strip. So taking moments about $x$-axis leads to the formula:

$$y = \frac{\int_a^b \frac{y^2}{2}\rho \, dx}{\int_a^b y\rho \, dx}$$

Now suppose the lamina described above is rotated about the $x$-axis to form a solid of revolution. An element would consist of a disc, centre on the $x$-axis, radius $y$, and thickness $\delta x$.

The volume of each element is $\pi y^2 \delta x$, leading to:

$m_i = \pi y^2 \rho \delta x$ where $\rho$ is density

Taking the limit as $\delta x \to 0$ gives (noting that $\pi$ cancels):

$$\bar{x} = \frac{\int_a^b xy^2\rho \, dx}{\int_a^b y^2\rho \, dx}$$

Again, note that $\rho$ cancels if the solid is uniform.

The above formulae should be known by students and can be quoted without proof. Use them to derive the standard results for a triangular lamina, semi-circular lamina, cone and hemisphere. Derivations of results for a sector, an arc of a circle, hollow cone and hemisphere should also be demonstrated.

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Problems may involve either showing results for standard shapes by integration or finding the centre of mass of a shape defined by a given curve such as in the question below.

A finite region bounded by the $x$-axis, the curve with equation $y = 2e^x$, the $y$-axis and the line $x = 1$ is rotated through one complete revolution about the $x$-axis to form a uniform solid. Use algebraic integration to

(a) show that the volume of the solid is $2\pi(e^2 - 1)$,

(b) find, in terms of e, the $x$-coordinate of the centre of mass of the solid.

(***Answer***: *x-coord* $= \frac{\pi(e^2+1)}{2\pi(e^2-1)} = \frac{e^2+1}{2(e^2-1)}$)

Note that 'algebraic integration' is required which means that approximate numerical methods including the use of a graphical calculator will not be credited.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students are sometimes let down in centre of mass questions by errors in basic algebra and integration; they are advised to show some intermediate steps in their calculations to avoid this.

As always, careful reading of the question is essential as in one question it was reported that 'large numbers of good candidates lost the final mark as they completely forgot that *coordinates* had been requested and so a $y$-coordinate was needed as well as the $x$-coordinate.'

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## NOTES

Formulae in terms of integrals can be quoted without proof.

Note, however, that it will be made clear in the question if a proof using integration is required; otherwise the results in the formula book can be used without proof.

Mostly algebraic integration (or requirement for exact answers) is specified and so calculator numerical methods are only useful for checking.

# A level Further Mathematics: Mechanics 3

| **5b. Centre of mass of composite bodies; Simple cases of equilibrium of rigid bodies (5.1) (5.2)** | **Teaching time** 4 hours |
|---|---|

## OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the centre of mass of a composite body by using 'moments' for the individual masses;
- recognise that the centre of mass lies on any axis of symmetry and on any plane of symmetry;
- be able to find the angle that any specified line on the shape makes with the vertical (or horizontal) when the object is freely suspended from a fixed point.

## TEACHING POINTS

We extend the work on 2D laminas and the method of creating a table of individual masses and centres of mass to 3D solids. The formulae for standard shapes can be quoted from the formula book.

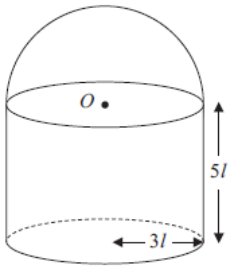
It is worth noting that the centre of mass lies on any axis of symmetry and on any plane of symmetry, and so it may be possible to write down one coordinate without showing any working.

In the table only the ratio of masses needs to be considered; for uniform solids this means any common factors such as densities, $\pi$ etc. do not need to be included, and the ratios can be cancelled down as far as possible.

An example of a typical configuration is shown below:

A solid consists of a uniform solid right circular cylinder of height $5l$ and radius $3l$ joined to a uniform solid hemisphere of radius $3l$. The plane face of the hemisphere coincides with a circular end of the cylinder and has centre, $O$. The density of the hemisphere is **twice** the density of the cylinder.

Find the distance of the centre of mass of the solid from $O$.




Notice that the composite body is non-uniform but the individual shapes are uniform so we use mass = density × volume taking into account one density is twice the other. Cancelling down the mass ratios considerably simplifies the working in the moments equation.

| | **Cylinder** | **Hemisphere** | **Whole solid** |
|---|---|---|---|
| **Mass ratio** | $\pi(3l)^2 \times 5l\rho$<br>5 | $\frac{2}{3}\pi(3l)^3 \times 2\rho$<br>4 | $81\pi l^3\rho$<br>9 |
| **Distance from *O*** | $\frac{5}{2}l$ | $-\frac{3}{8} \times 3l$ | $\bar{x}$ |

Moments equation: $5 \times \frac{5}{2}l - 4 \times -\frac{3}{8} \times 3l = 9\bar{x}$

$\bar{x} = \frac{8}{9}l$

Be aware that sometimes masses are specified (such as $m$ and $M$) in which case there is no need to use volumes.

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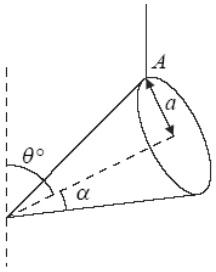
## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

If a composite body is suspended freely in equilibrium from a fixed point, then, as in the case of a lamina, the centre of mass ($G$) lies vertically below that point. Therefore the vertical is defined by the line joining the fixed point to $G$. It should be possible to use trigonometry to find the angle that any other specified line makes with the vertical (or horizontal).

This question is an example of a problem that uses this principle for 3D solids.

A uniform solid right circular cone has base radius $a$ and semi-vertical angle $\alpha$ where $\tan\alpha = \frac{1}{3}$. The cone is freely suspended by a string attached at a point $A$ on the rim of its base, and hangs in equilibrium with its axis of symmetry making an angle of $\theta°$ with the upward vertical.

Find, to one decimal place, the value of $\theta$.




Use the standard result that the centre of mass for the cone is $\frac{3}{4}h$ from the vertex along the axis and it lies vertically below $A$ when suspended, as shown.

(***Answer**: 53.1*)

Some problems may involve attaching a particle to a point on a solid to enable it to balance in a certain position. There are two alternative approaches:

either take moments about the point of suspension for the solid and particle separately

or recalculate the centre of mass for the new system and use the fact that this is directly below the point of suspension.

The second option sometimes proves easier when the geometry is particularly difficult. A clear diagram is essential.

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Good diagrams and clear layout is particularly important for these questions so students can clearly keep track of their working and see what is going on.

Using simplified ratios of the masses before writing down a moment equation often makes things much clearer and prevents some basic errors.

It is important that students use the information provided in the question. An examiner commented on one question: 'Some candidates ignored the information provided about the masses of the two parts and calculated their own "masses" by using volumes instead, usually assuming both sections to be solids.'

## NOTES

The equilibrium of a rigid body placed on a horizontal or inclined plane is considered in the next sub-unit.

# A level Further Mathematics: Mechanics 3

| **5c. Conditions for toppling/sliding (5.2)** | **Teaching time**<br>4 hours |
|---|---|

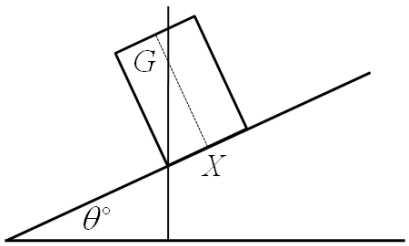
## OBJECTIVES

By the end of the sub-unit, students should:

- be able to identify the forces acting on a rigid body resting in equilibrium on a horizontal or inclined plane;
- understand and be able to apply the condition for a body in contact with a horizontal plane to be in equilibrium without toppling;
- understand and be able to apply the conditions for a body in contact with an inclined plane to be in equilibrium without either toppling or sliding.

## TEACHING POINTS

We want to consider the possible equilibrium of a body placed on a horizontal surface. If the body is resting in equilibrium then there will be no resultant moment about any point. Place a ball on a table. This will remain in equilibrium since the centre must lie above the point of contact (which means there is no resultant moment about any point). However, do the same with a hemispherical shape and demonstrate that there is only one position of equilibrium with the curved face in contact with the table (there will be a resultant moment about, for example, the point of contact if the centre of mass is not vertically above it).




Similarly, if a body is resting on an inclined plane, there must be no resultant moment about any point for equilibrium to be possible. In the example shown in the diagram the centre of mass ($G$) must be vertically above a point of contact for equilibrium to be possible. If $G$ is directly above the lowest point of contact then the body is said to be in limiting equilibrium (not to be confused with limiting equilibrium related to friction) or 'about to topple'.

There is, however, another condition to consider for equilibrium to be possible in the above case: will the object slide?

The condition $F \leq \mu R$ was seen as a condition for equilibrium in IAS Mechanics 1 and it is equally relevant here.

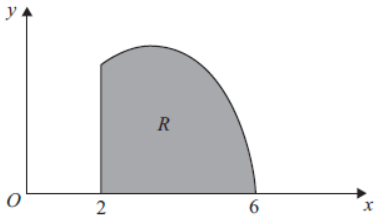
Place an ordinary rectangular eraser on its end on a book cover. Gradually raise the cover and very soon the eraser will topple. Then repeat but with the largest face resting on the cover. This is likely to start to slip before it topples.

# A level Further Mathematics: Mechanics 3

## OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Most problem solving for this topic will also involve finding the centre of mass of the body. This is illustrated in the example below which involves consideration of both toppling and sliding:

The shaded region $R$ is bounded by the curve with equation $y = \frac{1}{2}x\,(6 - x)$, the $x$-axis and the line $x = 2$. The unit of length on both axes is 1 cm. A uniform solid $P$ is formed by rotating $R$ through 360° about the $x$-axis.




a) Show that, to 3 significant figures, the centre of mass of $P$ is 1.42 cm from its plane face.

b) The uniform solid $P$ is placed with its plane face on an inclined plane which makes an angle $\theta$ with the horizontal. Given that the plane is sufficiently rough to prevent $P$ from sliding and that $P$ is on the point of toppling when $\theta = \alpha$, find the size of angle $\alpha$.

c) Given instead that $P$ is on the point of sliding down the plane when $\theta = \beta$ and that the coefficient of friction between $P$ and the plane is 0.3, find the size of angle $\beta$.

(***Answers***: *(b) 70.5, (c) 16.7 degrees*)

## COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students generally know that the centre of mass has to be directly above the lowest point of contact if the solid is about to topple, but many mistakes are made including using the wrong angle (by having the relevant trigonometric ratio upside down).

It is important that students appreciate the difference between limiting equilibrium when about to topple and limiting equilibrium when about to slide.

## NOTES

Clear diagrams are an essential part of this topic.