



AP[®]

INCLUDES

- ✓ Course framework
- ✓ Instructional section
- ✓ Sample exam questions

AP[®] Calculus AB and BC

COURSE AND EXAM DESCRIPTION

Effective
Fall 2020



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AP COURSE AND EXAM DESCRIPTIONS ARE UPDATED PERIODICALLY

Please visit [AP Central](#) to determine whether a more recent course and exam description is available.

What AP® Stands For

Thousands of Advanced Placement teachers have contributed to the principles articulated here. These principles are not new; they are, rather, a reminder of how AP already works in classrooms nationwide. The following principles are designed to ensure that teachers' expertise is respected, required course content is understood, and students are academically challenged and free to make up their own minds.

1. AP stands for clarity and transparency. Teachers and students deserve clear expectations. The Advanced Placement Program makes public its course frameworks and sample assessments. Confusion about what is permitted in the classroom disrupts teachers and students as they navigate demanding work.
2. AP is an unflinching encounter with evidence. AP courses enable students to develop as independent thinkers and to draw their own conclusions. Evidence and the scientific method are the starting place for conversations in AP courses.
3. AP opposes censorship. AP is animated by a deep respect for the intellectual freedom of teachers and students alike. If a school bans required topics from their AP courses, the AP Program removes the AP designation from that course and its inclusion in the AP Course Ledger provided to colleges and universities. For example, the concepts of evolution are at the heart of college biology, and a course that neglects such concepts does not pass muster as AP Biology.
4. AP opposes indoctrination. AP students are expected to analyze different perspectives from their own, and no points on an AP Exam are awarded for agreement with any specific viewpoint. AP students are not required to feel certain ways about themselves or the course content. AP courses instead develop students' abilities to assess the credibility of sources, draw conclusions, and make up their own minds.

As the AP English Literature course description states: "AP students are not expected or asked to subscribe to any one specific set of cultural or political values, but are expected to have the maturity to analyze perspectives different from their own and to question the meaning, purpose, or effect of such content within the literary work as a whole."

5. AP courses foster an open-minded approach to the histories and cultures of different peoples. The study of different nationalities, cultures, religions, races, and ethnicities is essential within a variety of academic disciplines. AP courses ground such studies in primary sources so that students can evaluate experiences and evidence for themselves.
6. Every AP student who engages with evidence is listened to and respected. Students are encouraged to evaluate arguments but not one another. AP classrooms respect diversity in backgrounds, experiences, and viewpoints. The perspectives and contributions of the full range of AP students are sought and considered. Respectful debate of ideas is cultivated and protected; personal attacks have no place in AP.
7. AP is a choice for parents and students. Parents and students freely choose to enroll in AP courses. Course descriptions are available online for parents and students to inform their choice. Parents do not define which college-level topics are suitable within AP courses; AP course and exam materials are crafted by committees of professors and other expert educators in each field. AP courses and exams are then further validated by the American Council on Education and studies that confirm the use of AP scores for college credits by thousands of colleges and universities nationwide.

The AP Program encourages educators to review these principles with parents and students so they know what to expect in an AP course. Advanced Placement is always a choice, and it should be an informed one. AP teachers should be given the confidence and clarity that once parents have enrolled their child in an AP course, they have agreed to a classroom experience that embodies these principles.

Contents

iv	Acknowledgments
1	About AP
5	AP Resources and Supports
6	Instructional Model
7	About the AP Calculus AB and BC Courses
7	College Course Equivalent
7	Prerequisites
8	Technology Needs and Use of Graphing Calculators

COURSE FRAMEWORK

10	Introduction
11	Mathematical Practices
13	Course Content
18	Course at a Glance
23	Unit Guides
25	Using the Unit Guides
27	UNIT 1: Limits and Continuity
49	UNIT 2: Differentiation: Definition and Fundamental Properties
65	UNIT 3: Differentiation: Composite, Implicit, and Inverse Functions
77	UNIT 4: Contextual Applications of Differentiation
89	UNIT 5: Analytical Applications of Differentiation
107	UNIT 6: Integration and Accumulation of Change
127	UNIT 7: Differential Equations
141	UNIT 8: Applications of Integration
161	UNIT 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY
175	UNIT 10: Infinite Sequences and Series BC ONLY

EXAM INFORMATION

198	Exam Overview
203	Sample AP Calculus AB and BC Exam Questions

SCORING GUIDELINES

217	Part A (AB or BC): Graphing Calculator Required
219	Part B (AB or BC): Calculator Not Permitted
221	Part A (BC only): Graphing Calculator Required
223	Part B (BC only): Calculator Not Permitted

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About AP

The Advanced Placement® Program (AP®) enables willing and academically prepared students to pursue college-level studies—with the opportunity to earn college credit, advanced placement, or both—while still in high school. Through AP courses in more than 40 subjects, each culminating in a challenging exam, students learn to think critically, construct solid arguments, and see many sides of an issue—skills that prepare them for college and beyond. Taking AP courses demonstrates to college admission officers that students have sought the most challenging curriculum available to them, and research indicates that students who score a 3 or higher on an AP Exam typically experience greater academic success in college and are more likely to earn a college degree than non-AP students. Each AP teacher's syllabus is evaluated and approved by faculty from some of the nation's leading colleges and universities, and AP Exams are developed and scored by college faculty and experienced AP teachers. Most four-year colleges and universities in the United States grant credit, advanced placement, or both on the basis of successful AP Exam scores—more than 3,300 institutions worldwide annually receive AP scores.

AP Course Development

In an ongoing effort to maintain alignment with best practices in college-level learning, AP courses and exams emphasize challenging, research-based curricula aligned with higher education expectations.

Individual teachers are responsible for designing their own curriculum for AP courses, selecting appropriate college-level readings, assignments, and resources. This course and exam description presents the content and skills that are the focus of the corresponding college course and that appear on the AP Exam. It also organizes the content and skills into a series of units that represent a sequence found in widely adopted college textbooks and that many AP teachers have told us they follow to focus their instruction. The intention of this publication is to respect teachers' time and expertise by providing a roadmap that they can modify and adapt to their local priorities and preferences. Moreover, by organizing the AP course content and skills into units, the AP Program is able to provide teachers and students with free formative assessments—Progress Checks—that teachers can assign throughout the year to measure student progress as they acquire content knowledge and develop skills.

Enrolling Students: Access, Opportunity, and Readiness

The AP Program welcomes all students willing to challenge themselves with college-level coursework and career preparation. We strongly encourage educators to invite students into AP classes, including students from ethnic, racial, socioeconomic, geographic, or other groups not broadly participating in a school's AP program. We believe that readiness for AP is attainable, and that educators can expand readiness by opening access to Pre-AP course work. We commit to support educators and communities in their efforts to make AP courses widely available, advancing students in their plans for college and careers.

Offering AP Courses: The AP Course Audit

The AP Program unequivocally supports the principle that each school implements its own curriculum that will enable students to develop the content understandings and skills described in the course framework.

While the unit sequence represented in this publication is optional, the AP Program does have a short list of curricular and resource requirements that must be fulfilled before a school can label a course “Advanced Placement” or “AP.” Schools wishing to offer AP courses must participate in the AP Course Audit, a process through which AP teachers’ course materials are reviewed by college faculty. The AP Course Audit was created to provide teachers and administrators with clear guidelines on curricular and resource requirements for AP courses and to help colleges and universities validate courses marked “AP” on students’ transcripts. This process ensures that AP teachers’ courses meet or exceed the curricular and resource expectations that college and secondary school faculty have established for college-level courses.

The AP Course Audit form is submitted by the AP teacher and the school principal (or designated administrator) to confirm awareness and understanding of the curricular and resource requirements. A syllabus or course outline, detailing how course requirements are met, is submitted by the AP teacher for review by college faculty.

Please visit the [AP Course Audit](#) website for more information to support the preparation and submission of materials for the AP Course Audit.

How the AP Program Is Developed

The scope of content for an AP course and exam is derived from an analysis of hundreds of syllabi and course offerings of colleges and universities. Using this research and data, a committee of college faculty and expert AP teachers work within the scope of the corresponding college course to articulate what students should know and be able to do upon the completion of the AP course. The resulting course framework is the heart of this course and exam description and serves as a blueprint of the content and skills that can appear on an AP Exam.

The AP Test Development Committees are responsible for developing each AP Exam, ensuring the exam questions are aligned to the course framework. The AP Exam development process is a multiyear endeavor; all AP Exams undergo extensive review, revision, piloting, and analysis to ensure that questions are accurate, fair, and valid, and that there is an appropriate spread of difficulty across the questions.

Committee members are selected to represent a variety of perspectives and institutions (public and private, small and large schools and colleges), and a range of gender, racial/ethnic, and regional groups. A list of each subject’s current AP Test Development Committee members is available on [AP Central®](#).

Throughout AP course and exam development, College Board gathers feedback from various stakeholders in both secondary schools and higher education institutions. This feedback is carefully considered to ensure that AP courses and exams are able to provide students with a college-level learning experience and the opportunity to demonstrate their qualifications for advanced placement or college credit.

How AP Exams Are Scored

The exam scoring process, like the course and exam development process, relies on the expertise of both AP teachers and college faculty. While multiple-choice questions are scored by machine, the free-response questions and through-course performance assessments, as applicable, are scored by thousands of college faculty and expert AP teachers. Most are scored at the annual AP Reading, while a small portion is scored online. All AP Readers are thoroughly trained, and their work is monitored throughout the Reading for fairness and consistency. In each subject, a highly respected college faculty member serves as Chief Faculty Consultant and, with the help of AP Readers in leadership positions, maintains the accuracy of the scoring standards. Scores on the free-response questions and performance assessments are weighted and combined with the results of the computer-scored multiple-choice questions, and this raw score is converted into a composite AP score on a 1–5 scale.

AP Exams are **not** norm referenced or graded on a curve. Instead, they are criterion referenced, which means that every student who meets the criteria for an AP score of 2, 3, 4, or 5 will receive that score, no matter how many students that is. The criteria for the number of points students must earn on the AP Exam to receive scores of 3, 4, or 5—the scores that research consistently validates for credit and placement purposes—include:

- The number of points successful college students earn when their professors administer AP Exam questions to them
- Performance that researchers have found to be predictive of an AP student succeeding when placed into a subsequent higher-level college course
- The number of points college faculty indicate, after reviewing each AP question, that they expect is necessary to achieve each AP grade level

Using and Interpreting AP Scores

The extensive work done by college faculty and AP teachers in the development of the course and exam and throughout the scoring process ensures that AP Exam scores accurately represent students' achievement in the equivalent college course. Frequent and regular research studies establish the validity of AP scores as follows:

AP Score	Credit Recommendation	College Grade Equivalent
5	Extremely well qualified	A
4	Well qualified	A-, B+, B
3	Qualified	B-, C+, C
2	Possibly qualified	n/a
1	No recommendation	n/a

While colleges and universities are responsible for setting their own credit and placement policies, most private colleges and universities award credit and/or advanced placement for AP scores of 3 or higher. Additionally, most states in the U.S. have adopted statewide credit policies that ensure college credit for scores of 3 or higher at public colleges and universities. To confirm a specific college's AP credit/placement policy, use the search engine available on the [AP Credit Policy Search](#) page.

BECOMING AN AP READER

Each June, thousands of AP teachers and college faculty members from around the world gather for seven days in multiple locations to evaluate and score the free-response sections of the AP Exams. Ninety-eight percent of surveyed educators who took part in the AP Reading say it was a positive experience.

There are many reasons to consider becoming an AP Reader, including opportunities to:

- **Bring positive changes to the classroom:** Surveys show that the vast majority of returning AP Readers—both high school and college educators—make improvements to the way they teach or score because of their experience at the AP Reading.
- **Gain in-depth understanding of AP Exam and AP scoring standards:** AP Readers gain exposure to the quality and depth of the responses from the entire pool of AP Exam takers, and thus are better able to assess their students' work in the classroom.
- **Receive compensation:** AP Readers are compensated for their work during the Reading. Expenses, lodging, and meals are covered for Readers who travel.
- **Score from home:** AP Readers have online distributed scoring opportunities for certain subjects. Check the [AP Reader](#) site for details.
- **Earn Continuing Education Units (CEUs):** AP Readers earn professional development hours and CEUs that can be applied to PD requirements by states, districts, and schools.

HOW TO APPLY

Visit the [Become an AP Reader](#) site for eligibility requirements and to start the application process.

AP Resources and Supports

By completing a simple activation process at the start of the school year, teachers and students receive access to a robust set of classroom resources.

AP Classroom

AP Classroom is a dedicated online platform designed to support teachers and students throughout their AP experience. The platform provides a variety of powerful resources and tools to provide yearlong support to teachers and enable students to receive meaningful feedback on their progress.

UNIT GUIDES

Appearing in this publication and on AP Classroom, these planning guides outline all required course content and skills, organized into commonly taught units. Each unit guide suggests a sequence and pacing of content, scaffolds skill instruction across units, organizes content into topics, and provides tips on taking the AP Exam.

PROGRESS CHECKS

Formative AP questions for every unit provide feedback to students on the areas where they need to focus. Available online, Progress Checks measure knowledge and skills through multiple-choice questions with rationales to explain correct and incorrect answers, and free-response questions with scoring information.

REPORTS

Reports provides teachers with a one-stop shop for student results on all assignment types, including Progress Checks. Teachers can view class trends and see where students struggle with content and skills that will be assessed on the AP Exam. Students can view their own progress over time to improve their performance before the AP Exam.

QUESTION BANK

The Question Bank is a searchable library of all AP questions that teachers use to build custom practice for their students. Teachers can create and assign assessments with formative topic questions from practice or released AP Exams.

Class Section Setup and Enrollment

- Teachers and students sign into or create their College Board accounts.
- Teachers confirm that they have added the course they teach to their AP Course Audit account and have had it approved by their school's administrator.
- Teachers or AP coordinators, depending on who the school has decided is responsible, set up class sections so students can access AP resources and have exams ordered on their behalf.
- Students join class sections with a join code provided by their teacher or AP coordinator.
- Students will be asked for additional information upon joining their first class section.

Instructional Model

Integrating AP resources throughout the course can help students develop skills and conceptual understandings. The instructional model outlined below shows possible ways to incorporate AP resources into the classroom.

Plan

Teachers may consider the following approaches as they plan their instruction before teaching each unit.

- Review the overview at the start of each **unit guide** to identify essential questions, conceptual understandings, and skills for each unit.
- Use the **Unit at a Glance** table to identify related topics that build toward a common understanding, and then plan appropriate pacing for students.

Teach

When teaching, supporting resources could be used to build students' conceptual understanding and their mastery of skills.

- Use the topic pages in the **unit guides** to identify the required content.
- Integrate the content with a skill, considering any appropriate scaffolding.
- Use the available resources, including **AP Videos**, on the topic pages to bring a variety of assets into the classroom.

Assess

Teachers can measure student understanding of the content and skills covered in the unit and provide actionable feedback to students.

- While teaching each topic, use **AP Classroom** to assign students **topic questions** as a way to continuously check student understanding and provide just-in-time feedback.
- At the end of each unit, use **AP Classroom** to assign students the online **progress checks**, as homework or an in-class task.
- Provide question-level feedback to students through answer rationales; provide formative feedback using Reports.
- Create additional practice opportunities using the **Question Bank** and assign them through **AP Classroom**.

About the AP Calculus AB and BC Courses

AP Calculus AB and AP Calculus BC focus on students' understanding of calculus concepts and provide experience with methods and applications. Through the use of big ideas of calculus (e.g., modeling change, approximation and limits, and analysis of functions), each course becomes a cohesive whole, rather than a collection of unrelated topics. Both courses require students to use definitions and theorems to build arguments and justify conclusions.

The courses feature a multirepresentational approach to calculus, with concepts, results, and problems expressed graphically, numerically, analytically, and verbally. Exploring connections among these representations builds understanding of how calculus applies limits to develop important ideas, definitions, formulas, and theorems. A sustained emphasis on clear communication of methods, reasoning, justifications, and conclusions is essential. Teachers and students should regularly use technology to reinforce relationships among functions, to confirm written work, to implement experimentation, and to assist in interpreting results.

College Course Equivalent

AP Calculus AB is designed to be the equivalent of a first semester college calculus course devoted to topics in differential and integral calculus. AP Calculus BC is designed to be the equivalent to both first and second semester college calculus courses. AP Calculus BC applies the content and skills learned in AP Calculus AB to parametrically defined curves, polar curves, and vector-valued functions; develops additional integration techniques and applications; and introduces the topics of sequences and series.

Prerequisites

Before studying calculus, all students should complete the equivalent of four years of secondary mathematics designed for college-bound students: courses that should prepare them with a strong foundation in reasoning with algebraic symbols and working with algebraic structures. Prospective calculus students should take courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the composition of functions, the algebra of functions, and the graphs of functions.

Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts, and descriptors such as increasing and decreasing). Students should also know how the sine and cosine functions are

defined from the unit circle and know the values of the trigonometric functions at the numbers $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and their multiples. Students who take AP Calculus BC are expected to have basic familiarity with sequences and series, as well as some exposure to parametric and polar equations.

Technology Needs and Use of Graphing Calculators

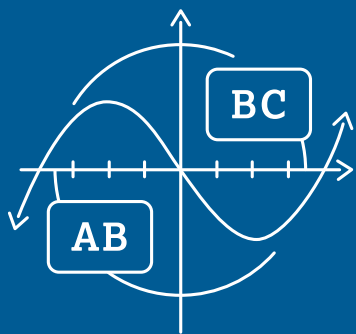
Technology should be used throughout the course as a tool to explore concepts. Specifically, the use of a graphing calculator is considered an integral part of the AP Calculus courses, and it is required on some portions of the AP Exam. Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the National Academy of Sciences (NAS) Board on Mathematical Sciences and Their Applications have strongly endorsed the use of calculators in mathematics instruction and testing.

A graphing calculator appropriate for use on the AP Exam is expected to have the built-in capabilities to plot the graph of a function within an arbitrary viewing window, find the zeros of functions (solve equations numerically), numerically calculate the derivative of a function, and numerically calculate the value of a definite integral. For free-response questions, when results are obtained using one of these required calculator capabilities, students are required to write the setup (e.g., the equation being solved, or the derivative or definite integral being evaluated) that leads to the solution, along with the result produced by the calculator. Use of other built-in calculator features or programs require the mathematical steps necessary to produce the results.

Graphing calculators are valuable tools for exploring multiple components of the mathematical practices, including using technology to develop conjectures, connecting concepts to their visual representations, solving problems, and critically interpreting and accurately reporting information. The AP Calculus courses also support the use of other technologies that are available to students and encourage teachers to incorporate technology into instruction in a variety of ways as a means of facilitating discovery and reflection.

Appropriate examples of graphing calculator use in AP Calculus include, but are certainly not limited to, the following:

- Zooming to reveal local linearity
- Constructing a table of values to conjecture a limit
- Developing a visual representation of Riemann sums approaching a definite integral
- Graphing Taylor polynomials to understand intervals of convergence for Taylor series
- Drawing a slope field and investigating how the choice of initial condition affects the solution to a differential equation



AP CALCULUS AB AND BC

Course Framework

Introduction

The course frameworks for AP Calculus AB and AP Calculus BC define content students must know and skills students must master in order to earn transferable, long-term understandings of calculus. The frameworks reflect a commitment to what college faculty value and mirror the curricula in corresponding college courses. Teachers may adjust the frameworks to meet state and local requirements.

The frameworks are organized into logical sequences, based on teacher input and commonly used textbooks. These sequences represent one reasonable learning pathway for each course, among many. Teachers may adjust the suggested sequencing of units or topics, although they will want to carefully consider how to account for such changes as they access course resources for planning, instruction, and assessment.

Balancing guidance and flexibility, this approach helps to prepare students for college credit and placement.

Mathematical Practices

The AP Calculus AB and BC mathematical practices describe what a student should be able to do while exploring course concepts. The table that follows presents these practices, which students should develop during the AP Calculus AB and AP Calculus BC courses. These practices are categorized into skills, which form the basis of the tasks on the AP Exam.

The unit guides later in this publication embed and spiral these skills throughout the course, providing teachers with one way to integrate the skills in the course content with sufficient repetition to prepare students to transfer those skills when taking the AP Exam. Course content may be paired with a variety of skills on the AP Exam.

Mathematical Practices

Practice 1	Practice 2	Practice 3	Practice 4
Implementing Mathematical Processes 1 Determine expressions and values using mathematical procedures and rules.	Connecting Representations 2 Translate mathematical information from a single representation or across multiple representations.	Justification 3 Justify reasoning and solutions.	Communication and Notation 4 Use correct notation, language, and mathematical conventions to communicate results or solutions.

SKILLS

1.A Identify the question to be answered or problem to be solved (*not assessed*).

1.B Identify key and relevant information to answer a question or solve a problem (*not assessed*).

1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., *Use the chain rule to find the derivative of a composite function*).

1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., *rate of change and accumulation*) or processes (e.g., *differentiation and its inverse process, anti-differentiation*) to solve problems.

1.E Apply appropriate mathematical rules or procedures, with and without technology.

1.F Explain how an approximated value relates to the actual value.

2.A Identify common underlying structures in problems involving different contextual situations.

2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

2.C Identify a re-expression of mathematical information presented in a given representation.

2.D Identify how mathematical characteristics or properties of functions are related in different representations.

2.E Describe the relationships among different representations of functions and their derivatives.

3.A Apply technology to develop claims and conjectures (*not assessed*).

3.B Identify an appropriate mathematical definition, theorem, or test to apply.

3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

3.D Apply an appropriate mathematical definition, theorem, or test.

3.E Provide reasons or rationales for solutions and conclusions.

3.F Explain the meaning of mathematical solutions in context.

3.G Confirm that solutions are accurate and appropriate.

4.A Use precise mathematical language.

4.B Use appropriate units of measure.

4.C Use appropriate mathematical symbols and notation (e.g., *Represent a derivative using $f'(x)$, y' , and $\frac{dy}{dx}$*).

4.D Use appropriate graphing techniques.

4.E Apply appropriate rounding procedures.

Course Content

Based on the Understanding by Design® (Wiggins and McTighe) model, this course framework provides a clear and detailed description of the course requirements necessary for student success. The framework specifies what students must know, be able to do, and understand, with a focus on big ideas that encompass core principles, theories, and processes of the discipline. The framework also encourages instruction that prepares students for advanced coursework in mathematics or other fields engaged in modeling change (e.g., pure sciences, engineering, or economics) and for creating useful, reasonable solutions to problems encountered in an ever-changing world.

BIG IDEAS

The big ideas serve as the foundation of the course and allow students to create meaningful connections among concepts. They are often abstract concepts or themes that become threads that run throughout the course. Revisiting the big ideas and applying them in a variety of contexts allows students to develop deeper conceptual understanding. Below are the big ideas of the course and a brief description of each.

BIG IDEA 1: CHANGE (CHA)

Using derivatives to describe rates of change of one variable with respect to another or using definite integrals to describe the net change in one variable over an interval of another allows students to understand change in a variety of contexts. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus—a central idea in AP Calculus.

BIG IDEA 2: LIMITS (LIM)

Beginning with a discrete model and then considering the consequences of a limiting case allows us to model real-world behavior and to discover and understand important ideas, definitions, formulas, and theorems in calculus: for example, continuity, differentiation, integration, and series **BC ONLY**.

BIG IDEA 3: ANALYSIS OF FUNCTIONS (FUN)

Calculus allows us to analyze the behaviors of functions by relating limits to differentiation, integration, and infinite series and relating each of these concepts to the others.

UNITS

The course content is organized into commonly taught units. The units have been arranged in a logical sequence frequently found in many college courses and textbooks.

The eight units in AP Calculus AB and ten units in AP Calculus BC, and their weighting on the multiple-choice section of the AP Exam, are listed on the following page.

Pacing recommendations at the unit level and on the Course at a Glance provide suggestions for how teachers can teach the required course content and administer the Progress Checks.

The suggested class periods are based on a schedule in which the class meets five days a week for 45 minutes each day. While these recommendations have been made to aid planning, teachers are of course free to adjust the pacing based on the needs of their students, alternate schedules (e.g., block scheduling), or their school's academic calendar.

TOPICS

Each unit is broken down into teachable segments called topics. The topic pages (starting on p. 34) contain the required content for each topic. Although most topics can be taught in one or two class periods, teachers should pace the course to suit the needs of their students and school.

Exam Weighting for the Multiple-Choice Section of the AP Exam		
Units	Exam Weighting (AB)	Exam Weighting (BC)
Unit 1: Limits and Continuity	10–15%	5–10%
Unit 2: Differentiation: Definition and Fundamental Properties	10–15%	5–10%
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	5–10%	5–10%
Unit 4: Contextual Applications of Differentiation	10–15%	5–10%
Unit 5: Analytical Applications of Differentiation	15–20%	10–15%
Unit 6: Integration and Accumulation of Change	15–20%	15–20%
Unit 7: Differential Equations	5–10%	5–10%
Unit 8: Applications of Integration	10–15%	5–10%
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY		10–15%
Unit 10: Infinite Sequences and Series BC ONLY		15–20%

Spiraling the Big Ideas

The following table shows how the big ideas spiral across units.

Big Ideas	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5
	Limits and Continuity	Differentiation: Definition and Fundamental Properties	Differentiation: Composite, Implicit, and Inverse Functions	Contextual Applications of Differentiation	Analytical Applications of Differentiation
Change CHA					
Limits LIM					
Analysis of Functions FUN					

Spiraling the Big Ideas (cont'd)

Big Ideas	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
	Integration and Accumulation of Change	Differential Equations	Applications of Integration	Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY	Infinite Sequences and Series BC ONLY
Change CHA	✓		✓	✓	
Limits LIM	✓				✓
Analysis of Functions FUN	✓	✓		✓	

Course at a Glance

Plan

The Course at a Glance provides a useful visual organization of the AP Calculus AB and AP Calculus BC curricular components, including:

- Sequence of units, along with approximate weighting and suggested pacing. Please note, pacing is based on 45-minute class periods, meeting five days each week for a full academic year.
- Progression of topics within each unit.
- Spiraling of the big ideas and mathematical practices across units.

Teach

MATHEMATICAL PRACTICES

Mathematical practices spiral throughout the course.

- | | |
|--|-------------------------------------|
| 1 Implementing Mathematical Processes | 3 Justification |
| 2 Connecting Representations | 4 Communication and Notation |

BIG IDEAS

Big ideas spiral across topics and units.

- | | |
|-------------------|----------------------------------|
| CHA Change | FUN Analysis of Functions |
| LIM Limits | |

BC ONLY

The purple shading represents BC-only content.

Assess

Assign the Progress Checks—either as homework or in class—for each unit. Each Progress Check contains formative multiple-choice and free-response questions. The feedback from each Progress Check shows students the areas where they need to focus.

UNIT 1 Limits and Continuity	
AP EXAM WEIGHTING	10–15% AB 5–10% BC
CLASS PERIODS	~22–23 AB ~13–14 BC
Mathematical Practices	Topics
CHA 2	1.1 Introducing Calculus: Can Change Occur at an Instant?
LIM 2	1.2 Defining Limits and Using Limit Notation
LIM 2	1.3 Estimating Limit Values from Graphs
LIM 2	1.4 Estimating Limit Values from Tables
LIM 1	1.5 Determining Limits Using Algebraic Properties of Limits
LIM 1	1.6 Determining Limits Using Algebraic Manipulation
LIM 1	1.7 Selecting Procedures for Determining Limits
LIM 3	1.8 Determining Limits Using the Squeeze Theorem
LIM 2	1.9 Connecting Multiple Representations of Limits
LIM 3	1.10 Exploring Types of Discontinuities
LIM 3	1.11 Defining Continuity at a Point
LIM 1	1.12 Confirming Continuity over an Interval
LIM 1	1.13 Removing Discontinuities
LIM 3	1.14 Connecting Infinite Limits and Vertical Asymptotes
LIM 2	1.15 Connecting Limits at Infinity and Horizontal Asymptotes
FUN 3	1.16 Working with the Intermediate Value Theorem (IVT)

Progress Check 1

Multiple-choice: ~45 questions
Free-response: 3 questions (partial)

UNIT 2 Differentiation: Definition and Fundamental Properties	
AP EXAM WEIGHTING	10–15% AB 5–10% BC
CLASS PERIODS	~13–14 AB ~9–10 BC
Mathematical Practices	Topics
CHA 2	2.1 Defining Average and Instantaneous Rates of Change at a Point
CHA 1 4	2.2 Defining the Derivative of a Function and Using Derivative Notation
CHA 1	2.3 Estimating Derivatives of a Function at a Point
FUN 3	2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist
FUN 1	2.5 Applying the Power Rule
FUN 1	2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple
FUN LIM 1	2.7 Derivatives of $\cos x$, $\sin x$, e^x , and $\ln x$
FUN 1	2.8 The Product Rule
FUN 1	2.9 The Quotient Rule
FUN 1	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Progress Check 2

Multiple-choice: ~30 questions
Free-response: 3 questions (partial)

NOTE: Partial versions of the free-response questions are provided to prepare students for more complex, full questions that they will encounter on the AP Exam.

* indicates that Mathematical Practices 1 and 4 are covered by Topic 2.2.

UNIT 3

Differentiation: Composite, Implicit, and Inverse Functions

AP EXAM
WEIGHTING

5–10% AB

5–10% BC

CLASS PERIODS

~10–11 AB

~8–9 BC

Mathematical Practices	Topics
FUN 1	3.1 The Chain Rule
FUN 1	3.2 Implicit Differentiation
FUN 3	3.3 Differentiating Inverse Functions
FUN 1	3.4 Differentiating Inverse Trigonometric Functions
FUN 1	3.5 Selecting Procedures for Calculating Derivatives
FUN 1	3.6 Calculating Higher-Order Derivatives

Progress Check 3

Multiple-choice: ~15 questions
Free-response: 3 questions (partial/full)

UNIT 4

Contextual Applications of Differentiation

AP EXAM
WEIGHTING

10–15% AB

5–10% BC

CLASS PERIODS

~10–11 AB

~6–7 BC

Mathematical Practices	Topics
CHA 1	4.1 Interpreting the Meaning of the Derivative in Context
CHA 1	4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration
CHA 2	4.3 Rates of Change in Applied Contexts Other Than Motion
CHA 1	4.4 Introduction to Related Rates
CHA 3	4.5 Solving Related Rates Problems
CHA 1	4.6 Approximating Values of a Function Using Local Linearity and Linearization
LIM 3	4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Progress Check 4

Multiple-choice: ~15 questions
Free-response: 3 questions

UNIT 5

Analytical Applications of Differentiation

AP EXAM
WEIGHTING

15–20% AB

10–15% BC

CLASS PERIODS

~15–16 AB

~10–11 BC

Mathematical Practices	Topics
FUN 3	5.1 Using the Mean Value Theorem
FUN 3	5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points
FUN 2	5.3 Determining Intervals on Which a Function Is Increasing or Decreasing
FUN 3	5.4 Using the First Derivative Test to Determine Relative (Local) Extrema
FUN 1	5.5 Using the Candidates Test to Determine Absolute (Global) Extrema
FUN 2	5.6 Determining Concavity of Functions over Their Domains
FUN 3	5.7 Using the Second Derivative Test to Determine Extrema
FUN 2	5.8 Sketching Graphs of Functions and Their Derivatives
FUN 2	5.9 Connecting a Function, Its First Derivative, and Its Second Derivative
FUN 2	5.10 Introduction to Optimization Problems
FUN 3	5.11 Solving Optimization Problems
FUN 1 3	5.12 Exploring Behaviors of Implicit Relations

Progress Check 5

Multiple-choice: ~35 questions
Free-response: 3 questions

UNIT 6

Integration and Accumulation of Change

AP EXAM WEIGHTING

15–20% AB

15–20% BC

CLASS PERIODS

~18–20 AB

~15–16 BC

Mathematical Practices	Topics
CHA 4	6.1 Exploring Accumulations of Change
LIM 1	6.2 Approximating Areas with Riemann Sums
LIM 2	6.3 Riemann Sums, Summation Notation, and Definite Integral Notation
FUN 1	6.4 The Fundamental Theorem of Calculus and Accumulation Functions
FUN 2	6.5 Interpreting the Behavior of Accumulation Functions Involving Area
FUN 3	6.6 Applying Properties of Definite Integrals
FUN 3	6.7 The Fundamental Theorem of Calculus and Definite Integrals
FUN 4	6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation
FUN 1	6.9 Integrating Using Substitution
FUN 1	6.10 Integrating Functions Using Long Division and Completing the Square
FUN 1	6.11 Integrating Using Integration by Parts BC ONLY
FUN 1	6.12 Using Linear Partial Fractions BC ONLY
LIM 1	6.13 Evaluating Improper Integrals BC ONLY
FUN 1	6.14 Selecting Techniques for Antidifferentiation

Progress Check 6

Multiple-choice:

- ~25 questions (AB)
- ~35 questions (BC)

Free-response: 3 questions

UNIT 7

Differential Equations

AP EXAM WEIGHTING

5–10% AB

5–10% BC

CLASS PERIODS

~8–9 AB

~9–10 BC

Mathematical Practices	Topics
FUN 2	7.1 Modeling Situations with Differential Equations
FUN 3	7.2 Verifying Solutions for Differential Equations
FUN 2	7.3 Sketching Slope Fields
FUN 4	7.4 Reasoning Using Slope Fields
FUN 1	7.5 Approximating Solutions Using Euler's Method BC ONLY
FUN 1	7.6 Finding General Solutions Using Separation of Variables
FUN 1	7.7 Finding Particular Solutions Using Initial Conditions and Separation of Variables
FUN 3	7.8 Exponential Models with Differential Equations
FUN 3	7.9 Logistic Models with Differential Equations BC ONLY

Progress Check 7

Multiple-choice:

- ~15 questions (AB)
- ~20 questions (BC)

Free-response: 3 questions

UNIT 8

Applications of Integration

AP EXAM WEIGHTING

10–15% AB

5–10% BC

CLASS PERIODS

~19–20 AB

~13–14 BC

Mathematical Practices	Topics
CHA 1	8.1 Finding the Average Value of a Function on an Interval
CHA 1	8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals
CHA 3	8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts
CHA 4	8.4 Finding the Area Between Curves Expressed as Functions of x
CHA 1	8.5 Finding the Area Between Curves Expressed as Functions of y
CHA 2	8.6 Finding the Area Between Curves That Intersect at More Than Two Points
CHA 3	8.7 Volumes with Cross Sections: Squares and Rectangles
CHA 3	8.8 Volumes with Cross Sections: Triangles and Semicircles
CHA 3	8.9 Volume with Disc Method: Revolving Around the x - or y -Axis
CHA 2	8.10 Volume with Disc Method: Revolving Around Other Axes
CHA 4	8.11 Volume with Washer Method: Revolving Around the x - or y -Axis
CHA 2	8.12 Volume with Washer Method: Revolving Around Other Axes
CHA 3	8.13 The Arc Length of a Smooth, Planar Curve and Distance Traveled BC ONLY

Progress Check 8

Multiple-choice: ~30 questions

Free-response: 3 questions

UNIT 9

Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY

AP EXAM WEIGHTING

N/A AB

10–15% BC

CLASS PERIODS

N/A AB

~10–11 BC

Mathematical Practices

Topics

CHA 2	9.1 Defining and Differentiating Parametric Equations
CHA 1	9.2 Second Derivatives of Parametric Equations
CHA 1	9.3 Finding Arc Lengths of Curves Given by Parametric Equations
CHA 1	9.4 Defining and Differentiating Vector-Valued Functions
FUN 1	9.5 Integrating Vector-Valued Functions
FUN 1	9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions
FUN 2	9.7 Defining Polar Coordinates and Differentiating in Polar Form
CHA 3	9.8 Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve
CHA 3	9.9 Finding the Area of the Region Bounded by Two Polar Curves

Progress Check 9

Multiple-choice: ~25 questions
Free-response: 3 questions

UNIT 10

Infinite Sequences and Series BC ONLY

AP EXAM WEIGHTING

N/A AB

15–20% BC

CLASS PERIODS

N/A AB

~17–18 BC

Mathematical Practices

Topics

LIM 3	10.1 Defining Convergent and Divergent Infinite Series
LIM 3	10.2 Working with Geometric Series
LIM 3	10.3 The n th Term Test for Divergence
LIM 3	10.4 Integral Test for Convergence
LIM 3	10.5 Harmonic Series and p -Series
LIM 3	10.6 Comparison Tests for Convergence
LIM 3	10.7 Alternating Series Test for Convergence
LIM 3	10.8 Ratio Test for Convergence
LIM 3	10.9 Determining Absolute or Conditional Convergence
LIM 1	10.10 Alternating Series Error Bound
LIM 3 2	10.11 Finding Taylor Polynomial Approximations of Functions
LIM 1	10.12 Lagrange Error Bound
LIM 2	10.13 Radius and Interval of Convergence of Power Series
LIM 2	10.14 Finding Taylor or Maclaurin Series for a Function
LIM 3	10.15 Representing Functions as Power Series

Progress Check 10

Multiple-choice: ~45 questions
Free-response: 3 questions

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AP CALCULUS AB AND BC

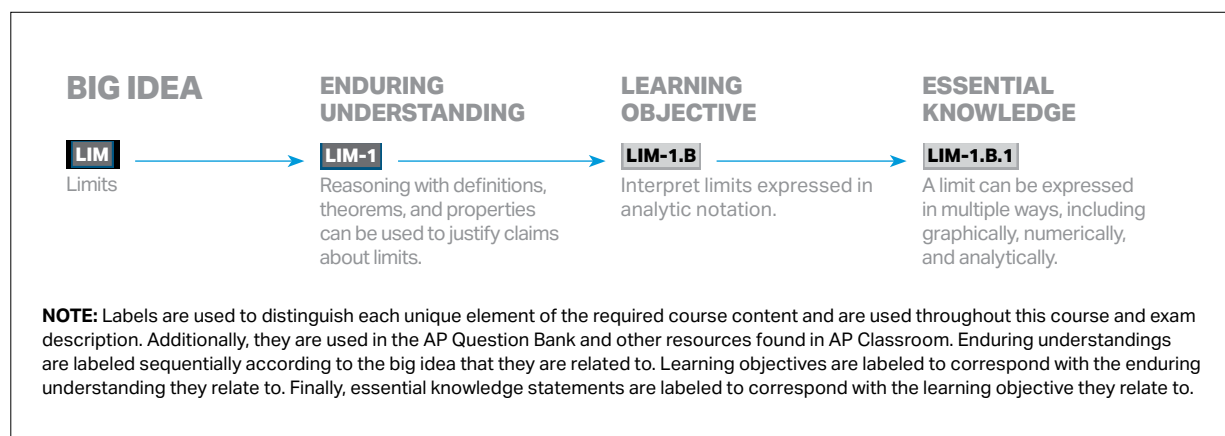
Unit Guides

Introduction

Designed with extensive input from the community of AP Calculus AB and AP Calculus BC educators, the unit guides offer teachers helpful guidance in building students' skills and knowledge. The suggested sequence was identified through a thorough analysis of the syllabi of highly effective AP teachers and the organization of typical college textbooks.

This unit structure respects new AP teachers' time by providing one possible sequence they can adopt and modify rather than having to build from scratch. An additional benefit is that these units enable the AP Program to provide interested teachers with formative assessments—the Progress Checks—that they can assign their students at the end of each unit to gauge progress toward success on the AP Exam. However, experienced AP teachers who are satisfied with their current course organization and exam results should feel no pressure to adopt these units, which comprise an optional sequence for this course.

REQUIRED COURSE CONTENT LABELING SYSTEM



Using the Unit Guides

The screenshot shows the Unit 1 guide for Limits and Continuity. At the top, it indicates the unit is for AP Calculus AB and BC, covering 10-15% of the course, with a weight of 5-10% and a duration of 22-23 class periods. The main title is "Limits and Continuity".

The guide is organized into three main sections:

- Developing Understanding**: This section introduces the subtle distinction between evaluating a function at a point and considering what value the function is approaching. It discusses the foundation for differentiation, integration, and infinite series.
- Building the Mathematical Practices**: This section describes specific skills within the practices that are appropriate to focus on in that unit. It notes that certain practices have been noted to indicate areas of emphasis for that unit.
- Preparing for the AP Exam**: This section provides helpful tips and common student misunderstandings identified from prior exam data. It emphasizes the importance of considering both specific content and skills related to each unit and building a coherent understanding of the whole.

On the left side, there are three "BIG IDEAS" listed:

- BIG IDEA 1 Change**: Can change occur at an instant?
- BIG IDEA 2 Limits**: How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?
- BIG IDEA 3 Analysis of Functions**: How do we close loopholes so that a conclusion about a function is always true?

UNIT OPENERS

Developing Understanding provides an overview that contextualizes and situates the key content of the unit within the scope of the course.

The **big ideas** serve as the foundation of the course and help develop understanding as they spiral throughout the course. The **essential questions** are thought-provoking questions that motivate students and inspire inquiry.

Building the Mathematical Practices describes specific skills within the practices that are appropriate to focus on in that unit. Certain practices have been noted to indicate areas of emphasis for that unit.

Preparing for the AP Exam provides helpful tips and common student misunderstandings identified from prior exam data.

Using the Unit Guides

UNIT 1 Limits and Continuity		
UNIT AT A GLANCE		
Enduring Understanding	Topic	Suggested Skills
CIA-1	1.1 Introducing Calculus: Can Change Occur at an Instant?	Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.2 Defining Limits and Using Limit Notation	Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.3 Estimating Limit Values from Graphs	Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.4 Estimating Limit Values from Tables	Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.5 Determining Limits Using Algebraic Properties of Limits	Apply appropriate mathematical rules or procedures, with and without technology.
LIM-1	1.6 Determining Limits Using Algebraic Manipulation	Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).
	1.7 Selecting Procedures for Determining Limits	Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).
	1.8 Determining Limits Using the Squeeze Theorem	Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.
	1.9 Connecting Multiple Representations of Limits	Identify a re-expression of mathematical information presented in a given representation.

The **Unit at a Glance** table shows the topics, related enduring understandings, and suggested skills. The class periods column has been left blank so that teachers can customize the time they spend on each topic.

The **suggested skills** for each topic show possible ways to link the content in that topic to specific AP Calculus AB and/or BC skills. The individual skills have been thoughtfully chosen in a way that scaffolds the skills throughout the course. The questions on the Progress Checks are based on this pairing. However, AP Exam questions can pair the content with any of the skills.

UNIT 1 Limits and Continuity		
SAMPLE INSTRUCTIONAL ACTIVITIES		
The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.		
Activity	Topic	Sample Activity
1	1.2	Notation Read Aloud Begin by writing a limit expression in analytical form (e.g., $\lim_{x \rightarrow 0} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	Create Representations Present students with a limit expression in analytical form (e.g., $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$) and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	Work Backward Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simply using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	Discussion Groups Give each group of students a piecewise-defined function, a graph paper, and a list of x -values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x -value, and explain why. Ask students to take turns recording the group's conclusion for each x -value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	Think Aloud In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

The **Sample Instructional Activities** include optional activities that can help tie together the content and skills for a particular topic.

Using the Unit Guides

UNIT 10 Infinite Sequences and Series

SUGGESTED SKILLS
Justification
3.D Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES
• External Resource > Davidson Next

TOPIC 10.8
Ratio Test for Convergence

Required Course Content

ENDURING UNDERSTANDING
UM-7 Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE
UM-7.A Determine whether a series converges or diverges.
BC ONLY

ESSENTIAL KNOWLEDGE
UM-7.A.1 The ratio test is a method to determine whether a series of numbers converges or diverges.
BC ONLY

EXCLUSION STATEMENT
The ratio test is a method to determine whether a series of numbers converges or diverges. The integral test, comparison test, limit comparison test, alternating series test, and ratio test (b) convergence are assessed on the AP Calculus BC Exam. Other methods are not assessed on the exam. However, teachers may include additional methods in the course, if time permits.

188 AP Calculus AB and BC • Course Framework [RETURN TO CONTENTS](#)

TOPIC PAGES

The **suggested skills** offer possible skills to pair with the topic.

Where possible, **available resources** are provided that might help address a particular topic. Some of these resources may have titles that refer to different topics, but those resources are listed for various units because they also model strategies that are applicable across multiple topics.

Enduring understandings are the long-term takeaways related to the big ideas that leave a lasting impression on students.

Essential knowledge statements define the required content knowledge associated with each learning objective assessed on the AP Exam.

Learning objectives define what a student needs to be able to do with content knowledge to progress through the course.

Exclusion statements identify topics that may be covered in a first-year college calculus course but are not assessed on the AP Calculus AB or BC Exam. Although these topics are not assessed, the AP Calculus courses are designed to support teachers who wish to introduce these topics to students.

UNIT 10 Infinite Sequences and Series

SUGGESTED SKILLS
Justification
3.D Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES
• External Resource > Davidson Next

TOPIC 10.2
Working with Geometric Series

Required Course Content

ENDURING UNDERSTANDING
UM-7 Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE
UM-7.A Determine whether a series converges or diverges.
BC ONLY

ESSENTIAL KNOWLEDGE
UM-7.A.2 A geometric series is a series with a constant ratio between successive terms. **BC ONLY**

UM-7.A.4 If r is a real number and r is a real number such that $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. **BC ONLY**

182 AP Calculus AB and BC • Course Framework [RETURN TO CONTENTS](#)

BC ONLY

Page elements that have purple shading represent content that applies only to AP Calculus BC. Topics, learning objectives, essential knowledge statements, and content in the unit openers that apply only to AP Calculus BC are shaded and labeled **BC ONLY**. Note that all of Units 9 and 10 apply only to AP Calculus BC.

AP CALCULUS AB AND BC

UNIT 1

Limits and Continuity



10–15% AB

5–10% BC

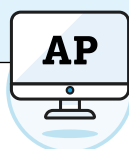
AP EXAM WEIGHTING



~22–23 AB

~13–14 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 1

Multiple-choice: ~45 questions

**Free-response: 3 questions
(partial)**

Limits and Continuity



BIG IDEA 1

Change **CHA**

- Can change occur at an instant?

BIG IDEA 2

Limits **LIM**

- How does knowing the value of a limit, or that a limit does not exist, help you to make sense of interesting features of functions and their graphs?

BIG IDEA 3

Analysis of Functions **FUN**

- How do we close loopholes so that a conclusion about a function is always true?

Developing Understanding

Limits introduce the subtle distinction between evaluating a function at a point and considering what value the function is approaching, if any, as x approaches a point. This distinction allows us to extend understanding of asymptotes and holes in graphs with formal definitions of continuity. Consider reviewing rational functions when introducing limits, rather than beginning the year with a full review of precalculus topics. Limits are the foundation for differentiation (Unit 2), integration (Unit 6), and infinite series (Unit 10) **BC ONLY**. They are the basis for important definitions and for theorems that are used to solve realistic problems involving change and to justify conclusions.

Building the Mathematical Practices

2.B 2.C 3.C 3.D

Mathematical information may be organized or presented graphically, numerically, analytically, or verbally. Mathematicians must be able to communicate effectively in all of these contexts and transition seamlessly from one representation to another. Limits lay the groundwork for students' ongoing development of skills associated with taking what is presented in a table, an equation, or a sentence and translating that information into a graph (or vice versa). Help students explicitly practice matching different representations that show the same information, focusing on building their comfort level with translating analytical and verbal representations. This will be instrumental to their development of proficiency in this practice. The use of graphing calculators to help students explore these connections is strongly encouraged.

Mathematicians also explain reasoning and justify conclusions using definitions, theorems, and tests. A common student misunderstanding is that they don't need to write relevant given information before drawing the conclusion of a theorem. In Unit 1, students should be given explicit instruction and time to practice "connecting the dots" by first demonstrating that all conditions or hypotheses have been met and then drawing the conclusion.

Preparing for the AP Exam

This course is a full-year experience building toward mastery assessed using the AP Exam. Therefore, it is important to consider both specific content and skills related to each unit and to build a coherent understanding of the whole. After studying Unit 1, students will be expected to evaluate or estimate limits presented graphically, numerically, analytically, or verbally. To avoid missed opportunities to earn points on the AP Exam, students need opportunities to consistently practice using correct mathematical notation and presenting set-ups and appropriately rounded answers when using a calculator. Two sections of the exam do not allow calculator use. Some questions on the other two sections require it. From the first unit onward, emphasize the importance of hypotheses for theorems. Explore why each hypothesis is needed in order to ensure that the conclusion follows. Students should establish the practice of explicitly verifying that a theorem's hypotheses are satisfied before applying the theorem.

UNIT AT A GLANCE

Enduring Understanding	Class Periods	
	Topic	Suggested Skills
CHA-1	1.1 Introducing Calculus: Can Change Occur at an Instant?	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.2 Defining Limits and Using Limit Notation	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
LIM-1	1.3 Estimating Limit Values from Graphs	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.4 Estimating Limit Values from Tables	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
	1.5 Determining Limits Using Algebraic Properties of Limits	1.E Apply appropriate mathematical rules or procedures, with and without technology.
	1.6 Determining Limits Using Algebraic Manipulation	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).
	1.7 Selecting Procedures for Determining Limits	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).
	1.8 Determining Limits Using the Squeeze Theorem	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.
	1.9 Connecting Multiple Representations of Limits	2.C Identify a re-expression of mathematical information presented in a given representation.

(Continued)

UNIT AT A GLANCE (cont'd)

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~22–23 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
LIM-2	1.10 Exploring Types of Discontinuities	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	1.11 Defining Continuity at a Point	3.C Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.	
	1.12 Confirming Continuity over an Interval	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.13 Removing Discontinuities	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	1.14 Connecting Infinite Limits and Vertical Asymptotes	3.D Apply an appropriate mathematical definition, theorem, or test.	
	1.15 Connecting Limits at Infinity and Horizontal Asymptotes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
FUN-1	1.16 Working with the Intermediate Value Theorem (IVT)	3.E Provide reasons or rationales for solutions or conclusions.	
 Go to AP Classroom to assign the Progress Check for Unit 1. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	1.2	Notation Read Aloud Begin by writing a limit expression in analytical form (e.g., $\lim_{x \rightarrow 0^-} x^3$), and then read the expression aloud to the class: "The limit of x cubed as x approaches 0 from the left." Do the same for 1–2 additional examples that use a variety of limit notations (e.g., the symbol for infinity). Then have students pair up and take turns reading aloud different limit expressions to one another.
2	1.3 1.4	Create Representations Present students with a limit expression in analytical form (e.g., $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$), and then have them translate that expression into a variety of representations: constructing a graph, creating a table of values, and writing it as a verbal expression. Then have students check their graphs and tables using technology.
3	1.7	Work Backward Present students with a set of limit problems. Rather than determining the given limits, have them make a list of the various strategies that would be used to determine the limits (e.g., factoring, multiplying by conjugate, and simplify using trigonometric identities). After confirming their list is complete, have students work in pairs to create and write limit problems, each requiring one of the listed strategies. Then have them swap problems with another pair of students to complete each other's problems.
4	1.11	Discussion Groups Give each group of students a piecewise-defined function, a graph paper, and a list of x -values. Have them graph the function, then discuss whether the function is continuous or discontinuous at each x -value, and explain why. Ask students to take turns recording the group's conclusion for each x -value. If continuous, have students discuss and show that all three continuity conditions are satisfied. If discontinuous, have students state which condition was not satisfied.
5	1.16	Think Aloud In small groups, have students discuss the Intermediate Value Theorem and share ideas about real-world applications (e.g., speed of your car and weight of your kitten). Have groups post their ideas on a classroom wall using sticky notes.

TOPIC 1.1

Introducing Calculus: Can Change Occur at an Instant?

SUGGESTED SKILLS

Connecting
Representations

2.B

Identify mathematical
information from graphical,
numerical, analytical, and/or
verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-1

Calculus allows us to generalize knowledge about motion to diverse problems involving change.

LEARNING OBJECTIVE

CHA-1.A

Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

ESSENTIAL KNOWLEDGE

CHA-1.A.1

Calculus uses limits to understand and model dynamic change.

CHA-1.A.2

Because an average rate of change divides the change in one variable by the change in another, the average rate of change is undefined at a point where the change in the independent variable would be zero.

CHA-1.A.3

The limit concept allows us to define instantaneous rate of change in terms of average rates of change.

SUGGESTED SKILLS

Connecting
Representations

2.B

Identify mathematical
information from graphical,
numerical, analytical, and/or
verbal representations.

TOPIC 1.2

Defining Limits and Using Limit Notation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.A

Represent limits analytically using correct notation.

LIM-1.B

Interpret limits expressed in analytic notation.

ESSENTIAL KNOWLEDGE

LIM-1.A.1

Given a function f , the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made arbitrarily close to R by taking x sufficiently close to c (but not equal to c). If the limit exists and is a real number, then the common notation is $\lim_{x \rightarrow c} f(x) = R$.

EXCLUSION STATEMENT

The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.

LIM-1.B.1

A limit can be expressed in multiple ways, including graphically, numerically, and analytically.

TOPIC 1.3

Estimating Limit Values from Graphs

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.1

The concept of a limit includes one sided limits.

LIM-1.C.2

Graphical information about a function can be used to estimate limits.

LIM-1.C.3

Because of issues of scale, graphical representations of functions may miss important function behavior.

LIM-1.C.4

A limit might not exist for some functions at particular values of x . Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

SUGGESTED SKILLS

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

ILLUSTRATIVE EXAMPLES

For LIM-1.C.4:

- $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$
- $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.
- $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exist.
- $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

AVAILABLE RESOURCES

- [AP Calculator Policy](#)

You can find the following related resource in AP Classroom:

- [Classroom Resource > Approximation](#)

SUGGESTED SKILLS

Connecting
Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

AVAILABLE RESOURCES

- [AP Calculator Policy](#)

You can find the following related resource in AP Classroom:

- Classroom Resource > Approximation

TOPIC 1.4

Estimating Limit Values from Tables

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.C

Estimate limits of functions.

ESSENTIAL KNOWLEDGE

LIM-1.C.5

Numerical information can be used to estimate limits.

TOPIC 1.5

Determining Limits Using Algebraic Properties of Limits

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

Required Course Content

ENDURING UNDERSTANDING**LIM-1**

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE**LIM-1.D**Determine the limits
of functions using limit
theorems.**ESSENTIAL KNOWLEDGE****LIM-1.D.1**One-sided limits can be determined analytically
or graphically.**LIM-1.D.2**Limits of sums, differences, products, quotients,
and composite functions can be found using
limit theorems.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

ILLUSTRATIVE EXAMPLES

- Factoring and dividing common factors of rational functions
- Multiplying by an expression involving the conjugate of a sum or difference in order to simplify functions involving radicals
- Using alternate forms of trigonometric functions

TOPIC 1.6

Determining Limits Using Algebraic Manipulation

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.1

It may be necessary or helpful to rearrange expressions into equivalent forms before evaluating limits.

TOPIC 1.7

Selecting Procedures for Determining Limits

This topic is intended to focus on the skill of selecting an appropriate procedure for determining limits. Students should be given opportunities to practice when and how to apply all learning objectives relating to determining limits.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

SUGGESTED SKILLS

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

ILLUSTRATIVE EXAMPLES

The squeeze theorem can be used to show

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \text{ and}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

TOPIC 1.8

Determining Limits Using the Squeeze Theorem

Required Course Content

ENDURING UNDERSTANDING

LIM-1

Reasoning with definitions, theorems, and properties can be used to justify claims about limits.

LEARNING OBJECTIVE

LIM-1.E

Determine the limits of functions using equivalent expressions for the function or the squeeze theorem.

ESSENTIAL KNOWLEDGE

LIM-1.E.2

The limit of a function may be found by using the squeeze theorem.

TOPIC 1.9

Connecting Multiple Representations of Limits

This topic is intended to focus on connecting representations. Students should be given opportunities to practice when and how to apply all learning objectives relating to limits and translating mathematical information from a single representation or across multiple representations.

SUGGESTED SKILLS

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- [AP Calculator Policy](#)

SUGGESTED SKILLS

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

AVAILABLE RESOURCES

- [AP Calculator Policy](#)

TOPIC 1.10

Exploring Types of Discontinuities

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.1

Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.

TOPIC 1.11

Defining Continuity at a Point

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.A

Justify conclusions about continuity at a point using the definition.

ESSENTIAL KNOWLEDGE

LIM-2.A.2

A function f is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \rightarrow c} f(x)$ exists, and $\lim_{x \rightarrow c} f(x) = f(c)$

SUGGESTED SKILLS

Justification

3.C

Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.

AVAILABLE RESOURCES

You can find the following related resource in the Online Teacher Community:

- AP Online Teacher Community Discussion > Video on Continuity

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

AVAILABLE RESOURCES

You can find the following
related resource in the Online
Teacher Community:

- AP Online Teacher
Community Discussion >
Video on Continuity

TOPIC 1.12

Confirming Continuity over an Interval

Required Course Content

ENDURING UNDERSTANDING**LIM-2**

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE**LIM-2.B**

Determine intervals
over which a function is
continuous.

ESSENTIAL KNOWLEDGE**LIM-2.B.1**

A function is continuous on an interval if the
function is continuous at each point in the
interval.

LIM-2.B.2

Polynomial, rational, power, exponential,
logarithmic, and trigonometric functions are
continuous on all points in their domains.

TOPIC 1.13

Removing Discontinuities

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

Required Course Content

ENDURING UNDERSTANDING**LIM-2**

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE**LIM-2.C**

Determine values of x or solve for parameters that make discontinuous functions continuous, if possible.

ESSENTIAL KNOWLEDGE**LIM-2.C.1**

If the limit of a function exists at a discontinuity in its graph, then it is possible to remove the discontinuity by defining or redefining the value of the function at that point, so it equals the value of the limit of the function as x approaches that point.

LIM-2.C.2

In order for a piecewise-defined function to be continuous at a boundary to the partition of its domain, the value of the expression defining the function on one side of the boundary must equal the value of the expression defining the other side of the boundary, as well as the value of the function at the boundary.

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 1.14

Connecting Infinite Limits and Vertical Asymptotes

Required Course Content

ENDURING UNDERSTANDING

LIM-2

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE

LIM-2.D

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE

LIM-2.D.1

The concept of a limit can be extended to include infinite limits.

LIM-2.D.2

Asymptotic and unbounded behavior of functions can be described and explained using limits.

TOPIC 1.15

Connecting Limits at Infinity and Horizontal Asymptotes

SUGGESTED SKILLS

Connecting
Representations**2.D**

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING**LIM-2**

Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.

LEARNING OBJECTIVE**LIM-2.D**

Interpret the behavior of functions using limits involving infinity.

ESSENTIAL KNOWLEDGE**LIM-2.D.3**

The concept of a limit can be extended to include limits at infinity.

LIM-2.D.4

Limits at infinity describe end behavior.

LIM-2.D.5

Relative magnitudes of functions and their rates of change can be compared using limits.

SUGGESTED SKILLS

Justification

3.E

Provide reasons or rationales for solutions or conclusions.

AVAILABLE RESOURCES

You can find the following related resource in the Online Teacher Community:

- Classroom Resource > Why We Use Theorem in Calculus

TOPIC 1.16

Working with the Intermediate Value Theorem (IVT)

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.A

Explain the behavior of a function on an interval using the Intermediate Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.A.1

If f is a continuous function on the closed interval $[a, b]$ and d is a number between $f(a)$ and $f(b)$, then the Intermediate Value Theorem guarantees that there is at least one number c between a and b , such that $f(c) = d$.

AP CALCULUS AB AND BC

UNIT 2

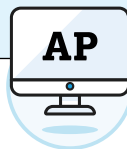
Differentiation: Definition and Fundamental Properties



10–15% AB
5–10% BC
AP EXAM WEIGHTING



~13–14 AB
~9–10 BC
CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 2

Multiple-choice: ~30 questions

**Free-response: 3 questions
(partial)**

Differentiation: Definition and Fundamental Properties



BIG IDEA 1

Change **CHA**

- How can a state determine the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) based on a model for the number of graduates as a function of the state's education budget?

BIG IDEA 2

Limits **LIM**

- Why do mathematical properties and rules for simplifying and evaluating limits apply to differentiation?

BIG IDEA 3

Analysis of Functions **FUN**

- If you knew that the rate of change in high school graduates at a particular level of public investment in education (in graduates per dollar) was a positive number, what might that tell you about the number of graduates at that level of investment?

Developing Understanding

Derivatives allow us to determine instantaneous rates of change. To develop understanding of how the definition of the derivative applies limits to average rates of change, create opportunities for students to explore average rates of change over increasingly small intervals. Graphing calculator explorations of how various operations affect slopes of tangent lines help students to make sense of basic rules and properties of differentiation. Encourage students to apply the order of operations as they select differentiation rules. Developing differentiation skills will allow students to model realistic instantaneous rates of change in Unit 4 and to analyze graphs in Unit 5.

Building the Mathematical Practices

1.E 2.B 4.C

Mathematicians know that a solution will only be as good as the procedure used to find it and that the difference between being correct and incorrect can often be traced to an arithmetic or procedural error. In other words, mathematicians know that the details matter. Students often find it difficult to apply mathematical procedures—including the rules of differentiation—with precision and accuracy. For example, students may drop important notation, such as a parenthesis, or misapply the product rule by taking the derivative of each factor separately and then multiplying those together. The content of Unit 2 is a foundational entry point for practicing the skill of applying mathematical procedures and learning to self-correct before common mistakes occur.

This is also an opportunity to revisit and reinforce the practice of connecting representations, as students will be seeing derivatives presented in analytical, numerical, graphical, and verbal representations. Students can practice by extracting information about the original function, f , from a graphical representation of f' . This can help prevent misunderstandings when examining the graph of a derivative (such as misinterpreting it as the graph of the original function instead).

Preparing for the AP Exam

Students should practice presenting clear mathematical structures that connect their work with definitions or theorems. For example, when asked to estimate the slope of the line tangent to a curve at a given point based on information provided in a table of values, students must present a difference quotient:

$C'(3.5) \approx \frac{C(4) - C(3)}{4 - 3} = \frac{12.8 - 11.2}{1}$. Failure to present this structure will cost students the point they might have earned, even with a correct numerical answer.

Similarly, to evaluate the derivative of $f(x) = u(x) \cdot v(x)$ at $x = 3$, students are expected to show the product rule structure, as in $f'(3) = u(3)v'(3) + v(3)u'(3)$, and import values. Finally, students need to present mathematical expressions evaluated on the calculator and use specified rounding procedures, typically rounding or truncating to three places after the decimal point. It is helpful to establish the habit of storing intermediate calculations in the calculator in order to avoid accumulation of rounding errors.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
CHA-2	2.1 Defining Average and Instantaneous Rates of Change at a Point	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	2.2 Defining the Derivative of a Function and Using Derivative Notation	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems. 4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y' , and $\frac{dy}{dx}$).	
	2.3 Estimating Derivatives of a Function at a Point	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-2	2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist	3.E Provide reasons or rationales for solutions and conclusions.	
FUN-3	2.5 Applying the Power Rule	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.6 Derivative Rules: Constant, Sum, Difference, and Constant Multiple	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3 LIM-3	2.7 Derivatives of $\cos x$, $\sin x$, e^x , and $\ln x$	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

(Continued)

UNIT AT A GLANCE (cont'd)

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~13–14 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-3	2.8 The Product Rule	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.9 The Quotient Rule	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	2.10 Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
 Go to AP Classroom to assign the Progress Check for Unit 2. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	2.1	Graph and Switch Present students with two or three functions and the graph of each function. Have each student choose a random derivative question and one function. Questions could include: Find the average rate of change on an interval, instantaneous rate of change at a point, derivative as a function, derivative value at a point, or equations for tangent or normal lines at a point. Have students answer their question and place their answer onto the function's graph. Then have students share their solutions with each other to give and receive feedback.
	2.2	
	2.3	
2	2.4	Match Mine Create cards containing graph images of functions with various continuous, discontinuous, differentiable, and nondifferentiable points or intervals. Provide each student in a pair with the same nine cards. Student A arranges their graphs in a 3×3 grid, which is not visible to Student B. Student A describes each of their graph's positions using information about continuity and differentiability to describe the graph. Based on the descriptions, Student B attempts to arrange their cards to match the grid of Student A.
3	2.5	Error Analysis Assign a function to each student. Ask them to find the function's derivative using one or more derivative rules. Allow them to check their answers. Ask half of the class to redo their work to include an error, thus having the wrong answer. Ask students to record their correct or incorrect work on a card. Mix up the cards and redistribute, having students determine if the answer is correct or incorrect. If incorrect, they should explain what error was made, and find the correct answer.
	2.6	
	2.7	
	2.8	
	2.9	
	2.10	
4	2.5	Graphic Organizer Provide students with colored paper, pens, and markers. Ask them to create a chart, a foldable card, or other creative method to organize all the derivative rules. For each rule, have them include the mathematical definition, examples, pictures, and helpful hints to understand and remember the rule.
	2.6	
	2.7	
	2.8	
	2.9	
	2.10	
5	2.8	Round Table Provide each student with the same worksheet containing four functions that require the product rule or quotient rule when finding the derivative. Then have students sit in groups of four. Each student determines the derivative of function No. 1, and then they pass their papers clockwise to the next student. Each student checks the first problem and, if necessary, discusses any mistakes with the previous student. Each student now completes function No. 2 on the paper, and the process continues until each student has their original paper back.
	2.9	

TOPIC 2.1

Defining Average and Instantaneous Rates of Change at a Point

SUGGESTED SKILLS

Connecting Representations

2.B

Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.A

Determine average rates of change using difference quotients.

CHA-2.B

Represent the derivative of a function as the limit of a difference quotient.

ESSENTIAL KNOWLEDGE

CHA-2.A.1

The difference quotients $\frac{f(a+h) - f(a)}{h}$ and $\frac{f(x) - f(a)}{x - a}$ express the average rate of change of a function over an interval.

CHA-2.B.1

The instantaneous rate of change of a function at $x = a$ can be expressed by $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ or $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists. These are equivalent forms of the definition of the derivative and are denoted $f'(a)$.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

Communication and
Notation

4.C

Use appropriate mathematical symbols and notation.

TOPIC 2.2

Defining the Derivative of a Function and Using Derivative Notation

Required Course Content

ENDURING UNDERSTANDING

CHA-2

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE

CHA-2.B

Represent the derivative of a function as the limit of a difference quotient.

CHA-2.C

Determine the equation of a line tangent to a curve at a given point.

ESSENTIAL KNOWLEDGE

CHA-2.B.2

The derivative of f is the function whose value at x is $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

CHA-2.B.3

For $y = f(x)$, notations for the derivative include $\frac{dy}{dx}$, $f'(x)$, and y' .

CHA-2.B.4

The derivative can be represented graphically, numerically, analytically, and verbally.

CHA-2.C.1

The derivative of a function at a point is the slope of the line tangent to a graph of the function at that point.

TOPIC 2.3

Estimating Derivatives of a Function at a Point

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**CHA-2**

Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.

LEARNING OBJECTIVE**CHA-2.D**

Estimate derivatives.

ESSENTIAL KNOWLEDGE**CHA-2.D.1**

The derivative at a point can be estimated from information given in tables or graphs.

CHA-2.D.2

Technology can be used to calculate or estimate the value of a derivative of a function at a point.

AVAILABLE RESOURCES

You can find the following related resources in AP Classroom:

- Classroom Resource > [Approximation](#)
- Classroom Resource > [Reasoning from Tabular Data](#)

SUGGESTED SKILLS

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

ILLUSTRATIVE EXAMPLES

For FUN-2.A.2:

- The left hand and right hand limits of the difference quotient are not equal, as in $f(x)=|x|$ at $x=0$.
- The tangent line is vertical and has no slope, as in $f(x)=\sqrt[3]{x}$ at $x=0$.

TOPIC 2.4

Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

Required Course Content

ENDURING UNDERSTANDING

FUN-2

Recognizing that a function's derivative may also be a function allows us to develop knowledge about the related behaviors of both.

LEARNING OBJECTIVE

FUN-2.A

Explain the relationship between differentiability and continuity.

ESSENTIAL KNOWLEDGE

FUN-2.A.1

If a function is differentiable at a point, then it is continuous at that point. In particular, if a point is not in the domain of f , then it is not in the domain of f' .

FUN-2.A.2

A continuous function may fail to be differentiable at a point in its domain.

TOPIC 2.5

Applying the Power Rule

SUGGESTED SKILLS

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.1

Direct application of the definition of the derivative and specific rules can be used to calculate the derivative for functions of the form $f(x) = x^r$.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

TOPIC 2.6

**Derivative Rules: Constant,
Sum, Difference, and
Constant Multiple**

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of
familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.2

Sums, differences, and constant multiples
of functions can be differentiated using
derivative rules.

FUN-3.A.3

The power rule combined with sum, difference,
and constant multiple properties can be used to
find the derivatives for polynomial functions.

TOPIC 2.7

Derivatives of $\cos x$, $\sin x$, e^x , and $\ln x$

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.A

Calculate derivatives of familiar functions.

ESSENTIAL KNOWLEDGE

FUN-3.A.4

Specific rules can be used to find the derivatives for sine, cosine, exponential, and logarithmic functions.

ENDURING UNDERSTANDING

LIM-3

Reasoning with definitions, theorems, and properties can be used to determine a limit.

LEARNING OBJECTIVE

LIM-3.A

Interpret a limit as a definition of a derivative.

ESSENTIAL KNOWLEDGE

LIM-3.A.1

In some cases, recognizing an expression for the definition of the derivative of a function whose derivative is known offers a strategy for determining a limit.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

TOPIC 2.8

The Product Rule

Required Course Content

ENDURING UNDERSTANDING**FUN-3**

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE**FUN-3.B**Calculate derivatives of
products and quotients of
differentiable functions.**ESSENTIAL KNOWLEDGE****FUN-3.B.1**Derivatives of products of differentiable
functions can be found using the product rule.

TOPIC 2.9

The Quotient Rule

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.2

Derivatives of quotients of differentiable functions can be found using the quotient rule.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

TOPIC 2.10

Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.B

Calculate derivatives of products and quotients of differentiable functions.

ESSENTIAL KNOWLEDGE

FUN-3.B.3

Rearranging tangent, cotangent, secant, and cosecant functions using identities allows differentiation using derivative rules.

AP CALCULUS AB AND BC

UNIT 3

Differentiation: Composite, Implicit, and Inverse Functions



5–10% AB

5–10% BC

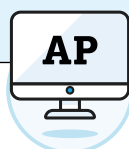
AP EXAM WEIGHTING



~10–11 AB

~8–9 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 3

Multiple-choice: ~15 questions

**Free-response: 3 questions
(partial/full)**

Differentiation: Composite, Implicit, and Inverse Functions

BIG IDEA 3

Analysis of Functions FUN

- If pressure experienced by a diver is a function of depth and depth is a function of time, how might we find the rate of change in pressure with respect to time?



Developing Understanding

In this unit, students learn how to differentiate composite functions using the chain rule and apply that understanding to determine derivatives of implicit and inverse functions. Students need to understand that for composite functions, y is a function of u while u

is a function of x . Leibniz notation for the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, accounts for these

relationships. Units analysis can strengthen the connection, as in $\frac{\text{psi}}{\text{min}} = \frac{\text{psi}}{\text{m}} \cdot \frac{\text{m}}{\text{min}}$.

Saying, “times the derivative of what’s inside,” every time we apply the chain rule

reminds students to avoid a common error. Mastering the chain rule is essential to success in all future units.

Building the Mathematical Practices

1.C 1.E 3.G

Identifying composite and implicit functions is a key differentiation skill. Students must recognize functions embedded in functions and be able to decompose composite functions into their “outer” and “inner” component functions. Misapplying the chain rule by forgetting to also differentiate the “inner” function or misidentifying the “inner” function are common errors. Provide sample responses that demonstrate these errors to help students be mindful of them in their own work. Reinforcing the chain rule structure sets the stage for Unit 6, when students learn the inverse of this process.

Students should continue to practice using correct notation and applying procedures accurately. Checking one another’s work, reviewing sample responses (with and without errors), and using technology to check calculations develop these skills. Emphasize that taking higher-order derivatives mirrors familiar differentiation processes (i.e., “function is to first derivative as first derivative is to second derivative”). Use questioning techniques such as, “What does this mean?” to help students develop a more solid conceptual understanding of higher-order differentiation.

Preparing for the AP Exam

Mastery of the chain rule and its applications is essential for success on the AP Exam. The chain rule will be the target of assessment for many questions and a necessary step along the way for others. One common error is not recognizing when the chain rule applies, especially in composite functions such as $\sin^2 x$, $\tan(2x - 1)$, and e^{x^2} . In

expressions like $\frac{y}{3y^2 - x}$, students must recognize that the chain rule applies to y

because y depends on x . When multiple rules apply, students may struggle with the order of operations. Offer mixed practice differentiating general functions using select values provided in tables and graphs. Focus on products, quotients, compositions, and inverses of functions, especially those with names other than f and g . Connecting graphs, tables, and algebraic reasoning builds understanding of differentiation of inverse functions.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods
			~10–11 CLASS PERIODS (AB) ~8–9 CLASS PERIODS (BC)
FUN-3	3.1 The Chain Rule	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.2 Implicit Differentiation	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.3 Differentiating Inverse Functions	3.G Confirm that solutions are accurate and appropriate.	
	3.4 Differentiating Inverse Trigonometric Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	3.5 Selecting Procedures for Calculating Derivatives	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
	3.6 Calculating Higher-Order Derivatives	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
 Go to AP Classroom to assign the Progress Check for Unit 3. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	3.1	Scavenger Hunt Place a card with a starter question somewhere in the classroom, for example, "Find the derivative of $f(x) = \sin(4x)$." Place another card in the room with the solution to that card plus another question: "Solution: $4\cos(4x)$. Next problem: Find the derivative of $f(x) = (\sin(x))^4$." Continue posting solution cards with new problems until the final card presents a problem whose solution is on the original starter card (note that this solution should be added to the starter card above).
2	3.1	Work Backward Provide a chain rule problem with several possible answers. Have students identify which piece of the original problem contributed its derivative to one of the factors in the answer they are looking at. For each possible answer, students should say whether the answer is correct. They should also identify which piece of the composition was differentiated incorrectly or skipped, or state which factor of that particular answer did not come from the original problem. Students often have trouble knowing when to stop differentiating when learning e^x or trigonometric functions, so make sure you include these types of problems.
3	3.2	Round Table In groups of four, each student has an identical paper with four different problems on it. Students complete the first problem on their paper and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Students rotate again and the process continues until each student has their own paper back.
4	3.4	Quiz-Quiz-Trade Give students a card containing a question and have them write the answer on the back. Students then circulate around the room and find a partner. One student quizzes the other by showing only the side of the card with the question on it, and then they reverse roles. They swap cards, find a new partner, quiz each other, and the process continues.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 3.1

The Chain Rule

Required Course Content

ENDURING UNDERSTANDING**FUN-3**

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE**FUN-3.C**

Calculate derivatives of compositions of differentiable functions.

ESSENTIAL KNOWLEDGE**FUN-3.C.1**

The chain rule provides a way to differentiate composite functions.

TOPIC 3.2

Implicit Differentiation

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.D

Calculate derivatives of implicitly defined functions.

ESSENTIAL KNOWLEDGE

FUN-3.D.1

The chain rule is the basis for implicit differentiation.

SUGGESTED SKILLS

Justification

3.G

Confirm that solutions are accurate and appropriate.

TOPIC 3.3

Differentiating Inverse Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.1

The chain rule and definition of an inverse function can be used to find the derivative of an inverse function, provided the derivative exists.

TOPIC 3.4

Differentiating Inverse Trigonometric Functions

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

ESSENTIAL KNOWLEDGE

FUN-3.E.2

The chain rule applied with the definition of an inverse function, or the formula for the derivative of an inverse function, can be used to find the derivatives of inverse trigonometric functions.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 3.5

Selecting Procedures for Calculating Derivatives

This topic is intended to focus on the skill of selecting an appropriate procedure for calculating derivatives. Students should be given opportunities to practice when and how to apply all learning objectives relating to calculating derivatives.

TOPIC 3.6

Calculating Higher -
Order Derivatives

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.F

Determine higher order
derivatives of a function.

ESSENTIAL KNOWLEDGE

FUN-3.F.1

Differentiating f' produces the second
derivative f'' , provided the derivative of f'
exists; repeating this process produces higher-
order derivatives of f .

FUN-3.F.2

Higher-order derivatives are represented with a
variety of notations. For $y = f(x)$, notations for
the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and
 y'' . Higher-order derivatives can be denoted
 $\frac{d^n y}{dx^n}$ or $f^{(n)}(x)$.

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AP CALCULUS AB AND BC

UNIT 4

Contextual Applications of Differentiation



10–15% AB

5–10% BC

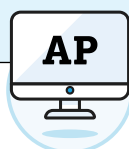
AP EXAM WEIGHTING



~10–11 AB

~6–7 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 4

Multiple-choice: ~15 questions

Free-response: 3 questions

Contextual Applications of Differentiation



BIG IDEA 1

Change **CHA**

- How are problems about position, velocity, and acceleration of a particle in motion over time structurally similar to problems about the volume of a rising balloon over an interval of heights, the population of London over the 14th century, or the metabolism of a dose of medicine over time?

BIG IDEA 2

Limits **LIM**

- Since certain indeterminate forms seem to actually approach a limit, how can we determine that limit, provided it exists?

Developing Understanding

Unit 4 begins by developing understanding of average and instantaneous rates of change in problems involving motion. The unit then identifies differentiation as a common underlying structure on which to build understanding of change in a variety of contexts. Students' understanding of units of measure often reinforces their understanding of contextual applications of differentiation. In problems involving related rates, identifying the independent variable common to related functions may help students to correctly apply the chain rule. When applying differentiation to determine limits of certain indeterminate forms using L'Hospital's rule, students must show that the rule applies.

Building the Mathematical Practices

1.D 1.F 2.A 3.F

Students will begin applying concepts from Units 2 and 3 to scenarios encountered in the world. Students often struggle to translate these verbal scenarios into the mathematical procedures necessary to answer the question. To solve these problems, students will need explicit instruction and intentional practice identifying key information, determining which procedure applies to the scenario presented (i.e., that “rates of change” indicate differentiation), stating what is changing and how, using correct units, and explaining what their answer means in the context of the scenario. Provide scenarios with different contexts but similar procedures so students begin to recognize and apply the reasoning behind those problem-solving decisions, rather than grasping at rules haphazardly.

Students must also be able to explain how an approximated value relates to the value it's intended to approximate. Students may not understand why they would use a tangent line approximation (i.e., linearization) rather than simply evaluating a function. Expose them to scenarios where an exact function value can't be calculated, and then ask them to determine whether a particular approximation is an overestimate or an underestimate of the function.

Preparing for the AP Exam

With contextual problems, emphasize careful reading for language such as, “find the rate of change.” This will help students understand the underlying structure of the problem, answer the question asked, and interpret solutions in context. Students should not use words like “velocity” when they mean the rate of change in income, for example, even though the underlying structure is the same.

Emphasize that students must verify that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ (or that both approach infinity) as a necessary first step before applying L'Hospital's Rule to

determine $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$. Students need to understand that $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are appropriate labels for

indeterminate forms but do not represent values in an equation. Therefore, it is incorrect

to write $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$, for example. Note that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \neq \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ when $\lim_{x \rightarrow a} g(x) = 0$.

Also emphasize that the conclusion of L'Hospital's rule features the ratio of the derivatives of the numerator and denominator, respectively, rather than the derivative of the ratio.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~10–11 CLASS PERIODS (AB) ~6–7 CLASS PERIODS (BC)
CHA-3	4.1 Interpreting the Meaning of the Derivative in Context	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	4.3 Rates of Change in Applied Contexts Other Than Motion	2.A Identify common underlying structures in problems involving different contextual situations.	
	4.4 Introduction to Related Rates	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	4.5 Solving Related Rates Problems	3.F Explain the meaning of mathematical solutions in context.	
	4.6 Approximating Values of a Function Using Local Linearity and Linearization	1.F Explain how an approximated value relates to the actual value.	
LIM-4	4.7 Using L'Hospital's Rule for Determining Limits of Indeterminate Forms	3.D Apply an appropriate mathematical definition, theorem, or test.	
 Go to AP Classroom to assign the Progress Check for Unit 4. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	4.1	Quickwrite Divide students into groups and give each group a context (outdoors, in a supermarket, in biology, in the government, at home, etc.). Students then write for a few minutes, listing things that are changing in that particular context.
2	4.2	Create Representations Provide verbal descriptions of a roller coaster ride: at time 0, velocity is 0 but about to become positive; at time 2, velocity is positive and increasing; at time 5, velocity is 0 and decreasing, etc. Have students graph position (from start), velocity, acceleration, speed, and then draw arrows at each point depicting whether their body would lean forward, backward, or not at all.
3	4.4	Marking the Text Have students read through a problem and highlight/underline the given quantities and directions in a problem, stating whether that information always applies or applies only at an instant.
4	4.5	Round Table Give students different related rates problems and a paper divided into five sections, titled as following: <ul style="list-style-type: none"> ▪ Draw a picture ▪ Equation ▪ Derivative ▪ Specific information used ▪ Interpretation Students first draw a picture of the situation and pass the papers clockwise. Students then critique the work in the previous section, complete the next section, and pass the papers again until all sections are completed.
5	4.6	Scavenger Hunt A starter question is posted in the room, for example, "Approximate the value . . ." Have students work through the problem to find the value and then look for that value at the top of another card posted in the room. Students then solve the problem on that card, for example, "Write the equation of the tangent line . . ." and look for that solution on a third card, etc. The solution to the last problem will be on the starter card.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

TOPIC 4.1

Interpreting the Meaning of the Derivative in Context

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.A

Interpret the meaning of a derivative in context.

ESSENTIAL KNOWLEDGE

CHA-3.A.1

The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable.

CHA-3.A.2

The derivative can be used to express information about rates of change in applied contexts.

CHA-3.A.3

The unit for $f'(x)$ is the unit for f divided by the unit for x .

TOPIC 4.2

Straight-Line Motion: Connecting Position, Velocity, and Acceleration

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

Required Course Content

ENDURING UNDERSTANDING**CHA-3**

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE**CHA-3.B**Calculate rates of change in
applied contexts.**ESSENTIAL KNOWLEDGE****CHA-3.B.1**The derivative can be used to solve rectilinear
motion problems involving position, speed,
velocity, and acceleration.

SUGGESTED SKILLS

Connecting
Representations

2.A

Identify common underlying
structures in problems involving
different contextual situations.

TOPIC 4.3

Rates of Change in Applied Contexts Other Than Motion

Required Course Content

ENDURING UNDERSTANDING**CHA-3**

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE**CHA-3.C**Interpret rates of change in
applied contexts.**ESSENTIAL KNOWLEDGE****CHA-3.C.1**The derivative can be used to solve problems
involving rates of change in applied contexts.

TOPIC 4.4

Introduction to Related Rates

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**CHA-3**

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE**CHA-3.D**

Calculate related rates in applied contexts.

ESSENTIAL KNOWLEDGE**CHA-3.D.1**

The chain rule is the basis for differentiating variables in a related rates problem with respect to the same independent variable.

CHA-3.D.2

Other differentiation rules, such as the product rule and the quotient rule, may also be necessary to differentiate all variables with respect to the same independent variable.

SUGGESTED SKILLS

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 4.5

Solving Related Rates Problems

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.E

Interpret related rates in applied contexts.

ESSENTIAL KNOWLEDGE

CHA-3.E.1

The derivative can be used to solve related rates problems; that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.

TOPIC 4.6

Approximating Values of a Function Using Local Linearity and Linearization

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.F

Explain how an approximated value relates to the actual value.

Required Course Content

ENDURING UNDERSTANDING**CHA-3**

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE**CHA-3.F**

Approximate a value on a curve using the equation of a tangent line.

ESSENTIAL KNOWLEDGE**CHA-3.F.1**

The tangent line is the graph of a locally linear approximation of the function near the point of tangency.

CHA-3.F.2

For a tangent line approximation, the function's behavior near the point of tangency may determine whether a tangent line value is an underestimate or an overestimate of the corresponding function value.

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 4.7

Using L'Hospital's Rule for Determining Limits of Indeterminate Forms

Required Course Content

ENDURING UNDERSTANDING

LIM-4

L'Hospital's Rule allows us to determine the limits of some indeterminate forms.

LEARNING OBJECTIVE

LIM-4.A

Determine limits of functions that result in indeterminate forms.

ESSENTIAL KNOWLEDGE

LIM-4.A.1

When the ratio of two functions tends to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in the limit, such forms are said to be indeterminate.

EXCLUSION STATEMENT

There are many other indeterminate forms, such as $\infty - \infty$, for example, but these will not be assessed on either the AP Calculus AB or BC Exam. However, teachers may include these topics, if time permits.

LIM-4.A.2

Limits of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.

AP CALCULUS AB AND BC

UNIT 5

Analytical Applications of Differentiation



15–20% AB

10–15% BC

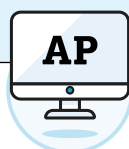
AP EXAM WEIGHTING



~15–16 AB

~10–11 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 5

Multiple-choice: ~35 questions

Free-response: 3 questions

Analytical Applications of Differentiation

BIG IDEA 3

Analysis of Functions FUN

- How might the Mean Value Theorem be used to justify a conclusion that you were speeding at some point on a certain stretch of highway, even without knowing the exact time you were speeding?
- What additional information is included in a sound mathematical argument about optimization that a simple description of an equivalent answer lacks?



Developing Understanding

In this unit, the superficial details of contextual applications of differentiation are stripped away to focus on abstract structures and formal conclusions. Reasoning with definitions and theorems establishes that answers and conclusions are more than conjectures; they have been analytically determined. As when students showed supporting work for answers in previous units, students will learn to present justifications for their conclusions about the behavior of functions over certain intervals or the locations of extreme values or points of inflection. The unit concludes this study of differentiation by applying abstract reasoning skills to justify solutions for realistic optimization problems.

Building the Mathematical Practices

1.E 2.D 2.E 3.E

The underlying processes of finding critical points and extrema are the foundation for the justifications students will write in this unit. Students should use calculators to graph a function and its derivatives to explore the related features of these graphs and confirm the results of their calculations.

Students often struggle with misinterpreting the characteristics of the graph of a derivative as though they are characteristics of the original function. Or, they use nonspecific language that conflates different functions (e.g., “it” rather than “ f ”). To prevent ongoing misconceptions, hold students accountable for extreme precision by having them practice matching graphs of functions to their derivatives and requiring them to explain their reasons to a peer.

Students also tend to rely on insufficient evidence or descriptions in their justifications, stating, for example, that “the graph of f is increasing because it’s going up.” This happens especially when examining derivative graphs on a calculator. Model calculus-based justifications (i.e., reasoning based on analysis of a derivative) both in discussion and in writing. Give students repeated opportunities to practice writing and revising their own justifications based on teacher feedback and feedback from their peers.

Preparing for the AP Exam

Sound reasoning must be accompanied by clear communication on the AP Exam. It may be helpful for students to use the language in the question as a starting point. Suppose a question asks, “Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.” A student who writes, “ g has neither a relative maximum nor a relative minimum at $x = 10$, because . . .,” has begun well. Similarly, given a graph of the derivative, f' , of a function, f , it is safer and easier for students to make arguments about f based directly on the graph of the derivative, as in, “ f is concave up on $a < x < b$ because the graph of f' is increasing on $a < x < b$.” Students are expected to refer to f , f' , and f'' by name, rather than by “it” or “the function,” which may leave the reader unsure of their intended meaning.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~15–16 CLASS PERIODS (AB) ~10–11 CLASS PERIODS (BC)
FUN-1	5.1 Using the Mean Value Theorem	3.E Provide reasons or rationales for solutions and conclusions.	
	5.2 Extreme Value Theorem, Global Versus Local Extrema, and Critical Points	3.E Provide reasons or rationales for solutions and conclusions.	
FUN-4	5.3 Determining Intervals on Which a Function is Increasing or Decreasing	2.E Describe the relationships among different representations of functions and their derivatives.	
	5.4 Using the First Derivative Test to Determine Relative (Local) Extrema	3.D Apply an appropriate mathematical definition, theorem, or test.	
	5.5 Using the Candidates Test to Determine Absolute (Global) Extrema	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	5.6 Determining Concavity of Functions over Their Domains	2.E Describe the relationships among different representations of functions and their derivatives.	
	5.7 Using the Second Derivative Test to Determine Extrema	3.D Apply an appropriate mathematical definition, theorem, or test.	
	5.8 Sketching Graphs of Functions and Their Derivatives	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.9 Connecting a Function, Its First Derivative, and Its Second Derivative	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	5.10 Introduction to Optimization Problems	2.A Identify common underlying structures in problems involving different contextual situations.	
	5.11 Solving Optimization Problems	3.F Explain the meaning of mathematical solutions in context.	
	5.12 Exploring Behaviors of Implicit Relations	1.E Apply appropriate mathematical rules or procedures, with and without technology. 3.E Provide reasons or rationales for solutions and conclusions.	
Go to AP Classroom to assign the Progress Check for Unit 5. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Suggested Activity
1	5.3	Critique Reasoning Arrange students in groups of four to six, provide them with a function's derivative (e.g., $g'(x) = 5x + 3$), and ask them to determine if $g(x)$ is increasing or decreasing at a specific x -value, for example, $x = -3$. Ask students to share the reasoning for their conclusion with classmates in their group. Members of the group can then provide feedback and suggestions.
2	5.4 5.7	Think-Pair-Share Provide students with a graph of f' and a graph of f'' . Ask them to identify relative extrema and practice writing justifications for relative extrema using the first or second derivative test. Once they've written their justification, ask them to pair with a partner and share their justifications. Students can then discuss similarities or differences in their justification wording.
3	5.5	Create a Plan Provide students with a function represented analytically on a closed interval. Ask them to discuss and write x -values that are viable candidates for absolute extrema. Once they have established the viable candidates, ask them to design a method for analyzing the behavior of the function's graph at the candidates and for identifying the extrema.
4	5.8 5.9	Predict and Confirm Provide students with the graph of a differentiable function, for example, $f(x) = x^3 - 4x^2 + 4x + 1$, but do not provide the rule for the function. Ask students to sketch a graph of the derivative of the function. Once students are done, reveal the rule for $f'(x)$. Ask students to calculate $f'(x)$, and use technology to graph $f'(x)$ and compare it to their sketched graph.

SUGGESTED SKILLS

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

You can find the following related resource in the Online Teacher Community:

- Classroom Resource > Why We Use Theorem in Calculus

TOPIC 5.1

Using the Mean Value Theorem

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.B

Justify conclusions about functions by applying the Mean Value Theorem over an interval.

ESSENTIAL KNOWLEDGE

FUN-1.B.1

If a function f is continuous over the interval $[a, b]$ and differentiable over the interval (a, b) , then the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.

TOPIC 5.2

Extreme Value Theorem, Global Versus Local Extrema, and Critical Points

Required Course Content

ENDURING UNDERSTANDING

FUN-1

Existence theorems allow us to draw conclusions about a function's behavior on an interval without precisely locating that behavior.

LEARNING OBJECTIVE

FUN-1.C

Justify conclusions about functions by applying the Extreme Value Theorem.

ESSENTIAL KNOWLEDGE

FUN-1.C.1

If a function f is continuous over the interval $[a, b]$, then the Extreme Value Theorem guarantees that f has at least one minimum value and at least one maximum value on $[a, b]$.

FUN-1.C.2

A point on a function where the first derivative equals zero or fails to exist is a critical point of the function.

FUN-1.C.3

All local (relative) extrema occur at critical points of a function, though not all critical points are local extrema.

SUGGESTED SKILLS

Justification

3.E

Provide reasons or rationales for solutions and conclusions.

AVAILABLE RESOURCES

You can find the following related resource in the Online Teacher Community:

- Classroom Resource > Why We Use Theorem in Calculus

You can find the following related resource in AP Classroom:

- Classroom Resource > Extrema

SUGGESTED SKILLS

Connecting
Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

TOPIC 5.3

Determining Intervals on Which a Function Is Increasing or Decreasing

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE**FUN-4.A.1**

The first derivative of a function can provide information about the function and its graph, including intervals where the function is increasing or decreasing.

TOPIC 5.4

Using the First Derivative Test to Determine Relative (Local) Extrema

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE**FUN-4.A.2**

The first derivative of a function can determine the location of relative (local) extrema of the function.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

TOPIC 5.5

Using the Candidates Test to Determine Absolute (Global) Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about
the behavior of a function
based on the behavior of its
derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.3

Absolute (global) extrema of a function on a
closed interval can only occur at critical points
or at endpoints.

TOPIC 5.6

Determining Concavity of Functions over Their Domains

SUGGESTED SKILLS

Connecting Representations

2.E

Describe the relationships among different representations of functions and their derivatives.

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE**FUN-4.A.4**

The graph of a function is concave up (down) on an open interval if the function's derivative is increasing (decreasing) on that interval.

FUN-4.A.5

The second derivative of a function provides information about the function and its graph, including intervals of upward or downward concavity.

FUN-4.A.6

The second derivative of a function may be used to locate points of inflection for the graph of the original function.

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 5.7

Using the Second Derivative Test to Determine Extrema

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.7

The second derivative of a function may determine whether a critical point is the location of a relative (local) maximum or minimum.

FUN-4.A.8

When a continuous function has only one critical point on an interval on its domain and the critical point corresponds to a relative (local) extremum of the function on the interval, then that critical point also corresponds to the absolute (global) extremum of the function on the interval.

TOPIC 5.8

Sketching Graphs of Functions and Their Derivatives

SUGGESTED SKILLS

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.A**

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE**FUN-4.A.9**

Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.

FUN-4.A.10

Graphical, numerical, and analytical information from f' and f'' can be used to predict and explain the behavior of f .

SUGGESTED SKILLS

Connecting
Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

TOPIC 5.9

Connecting a Function, Its First Derivative, and Its Second Derivative

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.A

Justify conclusions about the behavior of a function based on the behavior of its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.A.11

Key features of the graphs of f , f' , and f'' are related to one another.

TOPIC 5.10

Introduction to Optimization Problems

SUGGESTED SKILLS

Connecting
Representations

2.A

Identify common underlying structures in problems involving different contextual situations.

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.B**

Calculate minimum and maximum values in applied contexts or analysis of functions.

ESSENTIAL KNOWLEDGE**FUN-4.B.1**

The derivative can be used to solve optimization problems; that is, finding a minimum or maximum value of a function on a given interval.

SUGGESTED SKILLS

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 5.11

Solving Optimization Problems

Required Course Content

ENDURING UNDERSTANDING**FUN-4**

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE**FUN-4.C**

Interpret minimum and maximum values calculated in applied contexts.

ESSENTIAL KNOWLEDGE**FUN-4.C.1**

Minimum and maximum values of a function take on specific meanings in applied contexts.

TOPIC 5.12

Exploring Behaviors
of Implicit Relations

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

Justification

3.E

Provide reasons or rationales
for solutions and conclusions.

Required Course Content

ENDURING UNDERSTANDING

FUN-4

A function's derivative can be used to understand some behaviors of the function.

LEARNING OBJECTIVE

FUN-4.D

Determine critical points of
implicit relations.

FUN-4.E

Justify conclusions about
the behavior of an implicitly
defined function based on
evidence from its derivatives.

ESSENTIAL KNOWLEDGE

FUN-4.D.1

A point on an implicit relation where the first
derivative equals zero or does not exist is a
critical point of the function.

FUN-4.E.1

Applications of derivatives can be extended to
implicitly defined functions.

FUN-4.E.2

Second derivatives involving implicit
differentiation may be relations of x , y , and $\frac{dy}{dx}$.

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AP CALCULUS AB AND BC

UNIT 6

Integration and Accumulation of Change



15–20% AB

15–20% BC

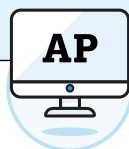
AP EXAM WEIGHTING



~18–20 AB

~15–16 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 6

Multiple-choice:

~25 questions (AB)

~35 questions (BC)

Free-response: 3 questions

Integration and Accumulation of Change



BIG IDEA 1

Change **CHA**

- Given information about a rate of population growth over time, how can we determine how much the population changed over a given interval of time?

BIG IDEA 2

Limits **LIM**

- If compounding more often increases the amount in an account with a given rate of return and term, why doesn't compounding continuously result in an infinite account balance, all other things being equal?

BIG IDEA 3

Analysis of Functions **FUN**

- How is integrating to find areas related to differentiating to find slopes?

Developing Understanding

This unit establishes the relationship between differentiation and integration using the Fundamental Theorem of Calculus. Students begin by exploring the contextual meaning of areas of certain regions bounded by rate functions. Integration determines accumulation of change over an interval, just as differentiation determines instantaneous rate of change at a point. Students need to understand that integration is a limiting case of a sum of products (areas) in the same way that differentiation is a limiting case of a quotient of differences (slopes). Future units will apply the idea of accumulation of change to a variety of realistic and geometric applications.

Building the Mathematical Practices

1.D 1.E 2.C 3.D

Students often struggle with the relationship between differentiation and integration. They think that integration is simply differentiation in reverse order. However, to apply the rules of integration correctly, students must think more strategically, taking into consideration how the “pieces” fit together. Students will need explicit guidance for choosing an appropriate antidifferentiation strategy that’s based on the underlying patterns in different categories of integrands (e.g., using u -substitution when they recognize that the integrand is a factor of the derivative of a composite function or using integration by parts for an integrand, $u dv$, that is related to a term in the derivative of the product uv **BC ONLY**).

Students also struggle with relating a symbolic limit of a Riemann sum to that limit expressed as a definite integral, because of the complexity of the expressions. To help students feel more comfortable working with these expressions, use explicit strategies, such as helping students to break complex expressions into familiar components, or matching expressions for a definite integral with the limit of a Riemann sum, and vice versa.

Preparing for the AP Exam

Students should be careful applying the chain rule, both when differentiating functions defined by integrals and when integrating using u -substitution. Students will need to recognize integrands that are factors of a chain rule derivative and will benefit from opportunities to practice u -substitution until the process is internalized. Students will additionally need to recognize integrands that suggest strategies such as integration by parts or partial fractions and should use mixed practice in preparation for the exam **BC ONLY**.

When using a calculator to evaluate a definite integral in a free-response question, students are expected to present the expression for the definite integral, including endpoints of integration, and an appropriately placed differential. When evaluating an integral without a calculator, students need to present an appropriate antiderivative and include a constant of integration with indefinite integrals. As always, students must be careful about parentheses usage and avoid writing strings of equal signs equating expressions that are not equal.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods
			~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
CHA-4	6.1 Exploring Accumulations of Change	4.B Use appropriate units of measure.	
LIM-5	6.2 Approximating Areas with Riemann Sums	1.F Explain how an approximated value relates to the actual value.	
	6.3 Riemann Sums, Summation Notation, and Definite Integral Notation	2.C Identify a re-expression of mathematical information presented in a given representation.	
FUN-5	6.4 The Fundamental Theorem of Calculus and Accumulation Functions	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	6.5 Interpreting the Behavior of Accumulation Functions Involving Area	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	6.6 Applying Properties of Definite Integrals	3.D Apply an appropriate mathematical definition, theorem, or test.	
	6.7 The Fundamental Theorem of Calculus and Definite Integrals	3.D Apply an appropriate mathematical definition, theorem, or test.	
FUN-6	6.8 Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y' , and $\frac{dy}{dx}$).	
	6.9 Integrating Using Substitution	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	6.10 Integrating Functions Using Long Division and Completing the Square	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

(Continued)

UNIT AT A GLANCE (cont'd)

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~18–20 CLASS PERIODS (AB) ~15–16 CLASS PERIODS (BC)
FUN-6	6.11 Integrating Using Integration by Parts BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	6.12 Integrating Using Linear Partial Fractions BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
LIM-6	6.13 Evaluating Improper Integrals BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-6	6.14 Selecting Techniques for Antidifferentiation	1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function).	
 Go to AP Classroom to assign the Progress Check for Unit 6. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	6.3	<p>Quickwrite</p> <p>Present the class with several examples of definite integrals set equal to Riemann sums in summation notation, for example, $\int_{-2}^5 (x^2 + 5) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{7}{n} \right) \left(\left(-2 + \frac{7}{n} i \right)^2 + 5 \right)$.</p> <p>Ask students to take five minutes to identify and write about all common elements between the two expressions and why they think the two expressions are equivalent. After finishing the five minutes, ask students to share their observations with the class.</p>
2	6.9	<p>Look For a Pattern</p> <p>Present students with several indefinite integrals and proposed, yet incorrect, antiderivatives, for example, $\int (5x + 2)^{20} dx = \frac{1}{21} (5x + 2)^{21} + C$. Ask them to check the antiderivatives by differentiating each and comparing to the original integrands. As students see that each antiderivative is incorrect, ask them to identify a pattern within the errors. Identifying this pattern will establish the foundation for integrating using substitution.</p>
3	6.10	<p>Odd One Out</p> <p>To help students select a strategy, form groups of four, presenting each student an indefinite integral whose integrand is rational. For each group, include one integrand that requires long division or completing the square. Ask students to decide if their example fits with the group. Identifying the odd one out will help students connect integrand form to the appropriate strategy.</p>

TOPIC 6.1

Exploring Accumulations of Change

SUGGESTED SKILLS

Communication and Notation

4.B

Use appropriate units of measure.

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.A

Interpret the meaning of areas associated with the graph of a rate of change in context.

ESSENTIAL KNOWLEDGE

CHA-4.A.1

The area of the region between the graph of a rate of change function and the x axis gives the accumulation of change.

CHA-4.A.2

In some cases, accumulation of change can be evaluated by using geometry.

CHA-4.A.3

If a rate of change is positive (negative) over an interval, then the accumulated change is positive (negative).

CHA-4.A.4

The unit for the area of a region defined by rate of change is the unit for the rate of change multiplied by the unit for the independent variable.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.F

Explain how an approximated
value relates to the actual value.

AVAILABLE RESOURCES

You can find the following
related resource in AP
Classroom:

- Classroom Resource >
Reasoning from Tabular
Data

TOPIC 6.2

Approximating Areas with Riemann Sums

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.A

Approximate a definite
integral using geometric and
numerical methods.

ESSENTIAL KNOWLEDGE

LIM-5.A.1

Definite integrals can be approximated for
functions that are represented graphically,
numerically, analytically, and verbally.

LIM-5.A.2

Definite integrals can be approximated using
a left Riemann sum, a right Riemann sum, a
midpoint Riemann sum, or a trapezoidal sum;
approximations can be computed using either
uniform or nonuniform partitions.

LIM-5.A.3

Definite integrals can be approximated using
numerical methods, with or without technology.

LIM-5.A.4

Depending on the behavior of a function, it
may be possible to determine whether an
approximation for a definite integral is an
underestimate or overestimate for the value of
the definite integral.

TOPIC 6.3

Riemann Sums, Summation Notation, and Definite Integral Notation

SUGGESTED SKILLS

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

Required Course Content

ENDURING UNDERSTANDING

LIM-5

Definite integrals can be approximated using geometric and numerical methods.

LEARNING OBJECTIVE

LIM-5.B

Interpret the limiting case of the Riemann sum as a definite integral.

LIM-5.C

Represent the limiting case of the Riemann sum as a definite integral.

ESSENTIAL KNOWLEDGE

LIM-5.B.1

The limit of an approximating Riemann sum can be interpreted as a definite integral.

LIM-5.B.2

A Riemann sum, which requires a partition of an interval I , is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition.

LIM-5.C.1

The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is the limit of Riemann sums as the widths of the subintervals approach 0. That is,

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i, \text{ where } n \text{ is}$$

the number of subintervals, Δx_i is the width of the i th subinterval, and x_i^* is a value in the i th subinterval.

LIM-5.C.2

A definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

ILLUSTRATIVE EXAMPLES

For FUN-5.A.1:

$$f(x) = \int_0^x e^{-t^2} dt.$$

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Functions Defined by Integrals

TOPIC 6.4

The Fundamental Theorem of Calculus and Accumulation Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-5

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE

FUN-5.A

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE

FUN-5.A.1

The definite integral can be used to define new functions.

FUN-5.A.2

If f is a continuous function on an interval

containing a , then $\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$, where x is in the interval.

TOPIC 6.5

Interpreting the Behavior of Accumulation Functions Involving Area

SUGGESTED SKILLS

Connecting Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

Required Course Content

ENDURING UNDERSTANDING**FUN-5**

The Fundamental Theorem of Calculus connects differentiation and integration.

LEARNING OBJECTIVE**FUN-5.A**

Represent accumulation functions using definite integrals.

ESSENTIAL KNOWLEDGE**FUN-5.A.3**

Graphical, numerical, analytical, and verbal representations of a function f provide information about the function g defined as

$$g(x) = \int_a^x f(t) dt.$$

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

TOPIC 6.6

Applying Properties of Definite Integrals

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.A

Calculate a definite integral using areas and properties of definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.A.1

In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.

FUN-6.A.2

Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.

FUN-6.A.3

The definition of the definite integral may be extended to functions with removable or jump discontinuities.

TOPIC 6.7

The Fundamental Theorem of Calculus and Definite Integrals

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.B

Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

ESSENTIAL KNOWLEDGE

FUN-6.B.1

An antiderivative of a function f is a function g whose derivative is f .

FUN-6.B.2

If a function f is continuous on an interval containing a , the function defined by $F(x) = \int_a^x f(t)dt$ is an antiderivative of f for x in the interval.

FUN-6.B.3

If f is continuous on the interval $[a, b]$ and F is an antiderivative of f , then $\int_a^b f(x)dx = F(b) - F(a)$.

SUGGESTED SKILLS

Communication and
Notation

4.C

Use appropriate mathematical
symbols
and notation.

TOPIC 6.8

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.C

Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

ESSENTIAL KNOWLEDGE

FUN-6.C.1 $\int f(x)dx$ is an indefinite integral of the function f and can be expressed as $\int f(x)dx = F(x) + C$, where $F'(x) = f(x)$ and C is any constant.**FUN-6.C.2**

Differentiation rules provide the foundation for finding antiderivatives.

FUN-6.C.3

Many functions do not have closed-form antiderivatives.

TOPIC 6.9

Integrating Using Substitution

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**FUN-6**

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE**FUN-6.D**

For integrands requiring substitution or rearrangements into equivalent forms:

- (a) Determine indefinite integrals.
- (b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE**FUN-6.D.1**

Substitution of variables is a technique for finding antiderivatives.

FUN-6.D.2

For a definite integral, substitution of variables requires corresponding changes to the limits of integration.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.10

Integrating Functions Using Long Division and Completing the Square

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.D

For integrands requiring substitution or rearrangements into equivalent forms:

- (a) Determine indefinite integrals.
- (b) Evaluate definite integrals.

ESSENTIAL KNOWLEDGE

FUN-6.D.3

Techniques for finding antiderivatives include rearrangements into equivalent forms, such as long division and completing the square.

TOPIC 6.11

Integrating Using Integration by Parts **BC ONLY**

SUGGESTED SKILLS

Implementing
Mathematical Processes**1.E**

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**FUN-6**

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE**FUN-6.E**

For integrands requiring integration by parts:

- (a) Determine indefinite integrals. **BC ONLY**
- (b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE**FUN-6.E.1**Integration by parts is a technique for finding antiderivatives. **BC ONLY**

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

TOPIC 6.12

Integrating Using Linear Partial Fractions **BC ONLY**

Required Course Content

ENDURING UNDERSTANDING

FUN-6

Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration.

LEARNING OBJECTIVE

FUN-6.F

For integrands requiring integration by linear partial fractions:

- (a) Determine indefinite integrals. **BC ONLY**
- (b) Evaluate definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-6.F.1

Some rational functions can be decomposed into sums of ratios of linear, nonrepeating factors to which basic integration techniques can be applied. **BC ONLY**

TOPIC 6.13

Evaluating Improper Integrals **BC ONLY**

SUGGESTED SKILLS

Implementing
Mathematical Processes**1.E**

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**LIM-6**

The use of limits allows us to show that the areas of unbounded regions may be finite.

LEARNING OBJECTIVE**LIM-6.A**Evaluate an improper integral or determine that the integral diverges. **BC ONLY****ESSENTIAL KNOWLEDGE****LIM-6.A.1**

An improper integral is an integral that has one or both limits infinite or has an integrand that is unbounded in the interval of integration.

BC ONLY**LIM-6.A.2**Improper integrals can be determined using limits of definite integrals. **BC ONLY**

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.C

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

TOPIC 6.14

Selecting Techniques for Antidifferentiation

This topic is intended to focus on the skill of selecting an appropriate procedure for antidifferentiation. Students should be given opportunities to practice when and how to apply all learning objectives relating to antidifferentiation.

AP CALCULUS AB AND BC

UNIT 7

Differential Equations



5–10% AB

5–10% BC

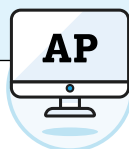
AP EXAM WEIGHTING



~8–9 AB

~9–10 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 7

Multiple-choice:

~15 questions (AB)

~20 questions (BC)

Free-response: 3 questions

Differential Equations

BIG IDEA 3

Analysis of Functions **FUN**

- How can we derive a model for the number of computers, C , infected by a virus, given a model for how fast the computers are being infected, $\frac{dC}{dt}$, at a particular time?



Developing Understanding

In this unit, students will learn to set up and solve separable differential equations. Slope fields can be used to represent solution curves to a differential equation and build understanding that there are infinitely many general solutions to a differential equation, varying only by a constant of integration. Students can locate a unique solution relevant to a particular situation, provided they can locate a point on the solution curve. By writing and solving differential equations leading to models for exponential growth and decay and logistic growth **BC ONLY**, students build understanding of topics introduced in Algebra II and other courses.

Building the Mathematical Practices

1.E 2.C 3.G 4.D

In this unit, students will translate mathematical information from one representation to another by matching equations and slope fields, rewriting verbal statements as differential equations, and sketching slope fields that match their symbolic representations. Provide students with explicit guidance on how to select an appropriate graphing technique. As students practice Euler's method, encourage them to transfer skills using tangent line approximations, rather than simply memorizing an algorithm **BC ONLY**.

Because the problems in this unit model real-world scenarios, help students to develop proficiency in transferring the mathematical procedures they've learned in "x's and y's" to equivalent environments with variable names other than x , y , and t . Using differentiation to confirm that solutions to differential equations are accurate and appropriate also helps students to develop an understanding of what it means to say that an equation is a solution to a differential equation.

Preparing for the AP Exam

Students need to practice setting up and solving contextual questions involving separable differential equations until the solution strategy becomes routine: separate variables, antidifferentiate both sides of the equation and add a constant of integration, use initial conditions to determine the constant of integration, and rearrange the resulting expression to complete the solution. Failure to separate variables or omitting the constant of integration severely limits the number of points a student can earn on the AP Exam. A common error in antidifferentiation is to assume that all differential equations involving fractions have logarithmic solutions, presumably because some do.

Students must learn to recognize the forms of differential equations resulting in exponential and logistic **BC ONLY** models. These may be used or interpreted without performing the derivation. Remind students that differential equations give us information about the derivative and may be used directly to find information about a slope or rate of change.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~8–9 CLASS PERIODS (AB) ~9–10 CLASS PERIODS (BC)
FUN-7	7.1 Modeling Situations with Differential Equations	2.C Identify a re-expression of mathematical information presented in a given representation.	
	7.2 Verifying Solutions for Differential Equations	3.G Confirm that solutions are accurate and appropriate.	
	7.3 Sketching Slope Fields	2.C Identify a re-expression of mathematical information presented in a given representation.	
	7.4 Reasoning Using Slope Fields	4.D Use appropriate graphing techniques.	
	7.5 Approximating Solutions Using Euler’s Method BC ONLY	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.6 Finding General Solutions Using Separation of Variables	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.7 Finding Particular Solutions Using Initial Conditions and Separation of Variables	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	7.8 Exponential Models with Differential Equations	3.G Confirm that solutions are accurate and appropriate.	
	7.9 Logistic Models with Differential Equations BC ONLY	3.F Explain the meaning of mathematical solutions in context.	
 Go to AP Classroom to assign the Progress Check for Unit 7. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	7.3	Match Mine Give student pairs a blank 3×3 game board and nine graphs of slope fields, each on a separate card. Some should be in terms of x only, some in terms of y only, and some in terms of x and y . Be sure to include at least one trigonometric function. Student A arranges the graphs on the grid without showing Student B and then describes the arrangement so Student B can try to match it on their own board.
2	7.6	Numbered Heads Together Have each student complete the same problem individually (e.g., $y' = 2xy^2$, $\frac{dy}{dx} = y^2 + 1$, or $3ydy = (x^2 + 1)dx$). Make sure to use a variety of notation in whatever problem you pick. Then have students compare answers and procedures within groups. Students fix any mistakes until they all agree on the same answer.
3	7.7 7.8	Collaborative Poster Assign each student a role within their group: <ul style="list-style-type: none"> • Separating the variables • Integrating both sides • Finding C • Writing the final particular solution Then distribute a free-response question to each group and have them work on their assigned roles to solve the problem together.

SUGGESTED SKILLS

Connecting
Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Differential Equations

TOPIC 7.1

Modeling Situations with Differential Equations

Required Course Content

ENDURING UNDERSTANDING**FUN-7**

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE**FUN-7.A**

Interpret verbal statements of problems as differential equations involving a derivative expression.

ESSENTIAL KNOWLEDGE**FUN-7.A.1**

Differential equations relate a function of an independent variable and the function's derivatives.

TOPIC 7.2

Verifying Solutions for Differential Equations

SUGGESTED SKILLS

Justification

3.G

Confirm that solutions are accurate and appropriate.

Required Course Content

ENDURING UNDERSTANDING**FUN-7**

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE**FUN-7.B**

Verify solutions to differential equations.

ESSENTIAL KNOWLEDGE**FUN-7.B.1**

Derivatives can be used to verify that a function is a solution to a given differential equation.

FUN-7.B.2

There may be infinitely many solutions to a differential equation.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Differential Equations

SUGGESTED SKILLS

Connecting
Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

You can find the following related resources in AP Classroom:

- Classroom Resource > Slope Fields
- Classroom Resource > Differential Equations

TOPIC 7.3

Sketching Slope Fields

Required Course Content

ENDURING UNDERSTANDING**FUN-7**

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE**FUN-7.C**

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE**FUN-7.C.1**

A slope field is a graphical representation of a differential equation on a finite set of points in the plane.

FUN-7.C.2

Slope fields provide information about the behavior of solutions to first-order differential equations.

TOPIC 7.4

Reasoning Using Slope Fields

SUGGESTED SKILLS

Communication and Notation

4.D

Use appropriate graphing techniques.

Required Course Content

ENDURING UNDERSTANDING**FUN-7**

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE**FUN-7.C**

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE**FUN-7.C.3**

Solutions to differential equations are functions or families of functions.

AVAILABLE RESOURCES

You can find the following related resources in AP Classroom:

- Classroom Resource > Slope Fields
- Classroom Resource > Differential Equations

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

You can find the following related resources in AP Classroom:

- Classroom Resource > Approximation
- Classroom Resource > Differential Equations

TOPIC 7.5

Approximating Solutions Using Euler's Method **BC ONLY**

Required Course Content

ENDURING UNDERSTANDING**FUN-7**

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE**FUN-7.C**

Estimate solutions to differential equations.

ESSENTIAL KNOWLEDGE**FUN-7.C.4**

Euler's method provides a procedure for approximating a solution to a differential equation or a point on a solution curve. **BC ONLY**

TOPIC 7.6

Finding General Solutions Using Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.D

Determine general solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.D.1

Some differential equations can be solved by separation of variables.

FUN-7.D.2

Antidifferentiation can be used to find general solutions to differential equations.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Differential Equations

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Differential Equations

TOPIC 7.7

Finding Particular Solutions Using Initial Conditions and Separation of Variables

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.E

Determine particular solutions to differential equations.

ESSENTIAL KNOWLEDGE

FUN-7.E.1

A general solution may describe infinitely many solutions to a differential equation. There is only one particular solution passing through a given point.

FUN-7.E.2

The function F defined by $F(x) = y_0 + \int_a^x f(t)dt$ is a particular solution to the differential equation $\frac{dy}{dx} = f(x)$, satisfying $F(a) = y_0$.

FUN-7.E.3

Solutions to differential equations may be subject to domain restrictions.

TOPIC 7.8

Exponential Models with Differential Equations

SUGGESTED SKILLS

Justification

3.G

Confirm that solutions are accurate and appropriate.

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.F

Interpret the meaning of a differential equation and its variables in context.

FUN-7.G

Determine general and particular solutions for problems involving differential equations in context.

ESSENTIAL KNOWLEDGE

FUN-7.F.1

Specific applications of finding general and particular solutions to differential equations include motion along a line and exponential growth and decay.

FUN-7.F.2The model for exponential growth and decay that arises from the statement "The rate of change of a quantity is proportional to the size of the quantity" is $\frac{dy}{dt} = ky$.**FUN-7.G.1**The exponential growth and decay model, $\frac{dy}{dt} = ky$, with initial condition $y = y_0$ when $t = 0$, has solutions of the form $y = y_0 e^{kt}$.

SUGGESTED SKILLS

Justification

3.F

Explain the meaning of mathematical solutions in context.

TOPIC 7.9

Logistic Models with Differential Equations **BC ONLY**

Required Course Content

ENDURING UNDERSTANDING

FUN-7

Solving differential equations allows us to determine functions and develop models.

LEARNING OBJECTIVE

FUN-7.H

Interpret the meaning of the logistic growth model in context. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-7.H.1

The model for logistic growth that arises from the statement "The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity" is $\frac{dy}{dt} = ky(a - y)$. **BC ONLY**

FUN-7.H.2

The logistic differential equation and initial conditions can be interpreted without solving the differential equation. **BC ONLY**

FUN-7.H.3

The limiting value (carrying capacity) of a logistic differential equation as the independent variable approaches infinity can be determined using the logistic growth model and initial conditions. **BC ONLY**

FUN-7.H.4

The value of the dependent variable in a logistic differential equation at the point when it is changing fastest can be determined using the logistic growth model and initial conditions. **BC ONLY**

AP CALCULUS AB AND BC

UNIT 8

Applications of Integration



10–15% AB

5–10% BC

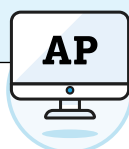
AP EXAM WEIGHTING



~19–20 AB

~13–14 BC

CLASS PERIODS



Remember to go to [AP Classroom](#) to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 8

Multiple-choice: ~30 questions

Free-response: 3 questions

Applications of Integration



BIG IDEA 1

Change **CHA**

- How is finding the number of visitors to a museum over an interval of time based on information about the rate of entry similar to finding the area of a region between a curve and the x -axis?

Developing Understanding

In this unit, students will learn how to find the average value of a function, model particle motion and net change, and determine areas, volumes, and lengths **BC ONLY** defined by the graphs of functions. Students will develop an understanding of integration that can be transferred across these and many other applications. Understanding that the area, volume, and length **BC ONLY** problems studied in this unit are limiting cases of Riemann sums of rectangle areas, prism volumes, or segment lengths **BC ONLY** saves students from memorizing a long list of seemingly unrelated formulas and develops meaningful understanding of integration.

Building the Mathematical Practices

1.D 2.D 3.D 4.C

As in Unit 4, students will need to practice interpreting verbal scenarios, extracting relevant mathematical information, selecting an appropriate procedure, and then applying that procedure correctly and interpreting their solution in the context of the problem. Now that students have been exposed to application problems involving both differentiation and antidifferentiation, some may struggle to determine which procedure is applicable. Walk students through different types of scenarios and explain the underlying reasons why some situations call for differentiation while others call for integration.

This unit also involves geometric applications of integration. When using the disc and washer methods, focusing on orientation (i.e., horizontal or vertical) will help students determine whether the “thickness” is with respect to x or y . Have students practice solving variations on these calculus-based geometry problems until they can decide which variable to integrate with respect to without prompting. Relating graphical representations to symbolic representations, such as Riemann sums and definite integrals, develops these skills and helps students to master the content.

Preparing for the AP Exam

On the AP Exam, students need to identify relevant information conveyed in various representations. Key words, such as “accumulation” or “net change,” help to identify mathematical structures and corresponding solution strategies. Some students confuse the average value and the average rate of change of a function on an interval. To alleviate confusion, first provide students with average value problems accompanied by relevant graphs and guide them to an understanding of why an average value may be less than, equal to, or greater than the midpoint of the range. Then review average rate of change problems from Unit 2 and present students with free-response questions that will allow them to practice distinguishing between average value and average rate of change problems.

In free-response questions, continue to require students to show supporting work by presenting a correct expression using appropriate notation and the mathematical structures of solutions, as in $V = \pi \int_1^4 \left[(f(x) - 3)^2 - (g(x) - 3)^2 \right] dx$, for example.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-4	8.1 Finding the Average Value of a Function on an Interval	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.2 Connecting Position, Velocity, and Acceleration of Functions Using Integrals	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
	8.3 Using Accumulation Functions and Definite Integrals in Applied Contexts	3.D Apply an appropriate mathematical definition, theorem, or test.	
CHA-5	8.4 Finding the Area Between Curves Expressed as Functions of x	4.C Use appropriate mathematical symbols and notation (e.g., Represent a derivative using $f'(x)$, y' , and $\frac{dy}{dx}$).	
	8.5 Finding the Area Between Curves Expressed as Functions of y	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	8.6 Finding the Area Between Curves That Intersect at More Than Two Points	2.B Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.	
	8.7 Volumes with Cross Sections: Squares and Rectangles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.8 Volumes with Cross Sections: Triangles and Semicircles	3.D Apply an appropriate mathematical definition, theorem, or test.	
	8.9 Volume with Disc Method: Revolving Around the x- or y-Axis	3.D Apply an appropriate mathematical definition, theorem, or test.	

(Continued)

UNIT AT A GLANCE (cont'd)

Enduring Understanding	Topic	Suggested Skills	Class Periods
			~19–20 CLASS PERIODS (AB) ~13–14 CLASS PERIODS (BC)
CHA-5	8.10 Volume with Disc Method: Revolving Around Other Axes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	8.11 Volume with Washer Method: Revolving Around the x - or y -Axis	4.E Apply appropriate rounding procedures.	
	8.12 Volume with Washer Method: Revolving Around Other Axes	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-6	8.13 The Arc Length of a Smooth, Planar Curve and Distance Traveled BC ONLY	3.D Apply an appropriate mathematical definition, theorem, or test.	
 Go to AP Classroom to assign the Progress Check for Unit 8. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	8.1	<p>Scavenger Hunt</p> <p>Post around the room 8–10 problem cards, each of which also includes a solution to a previous problem. Include average value problems with tables and functions; also include tables that have values from 0 to 16, for example, but where the average value is only on the interval from 0 to 12. Card 1 could say: Find the average value of $f(x) = \sin(x)$ on the interval $[0, \pi]$. A second card would have the answer to that card $\left(\frac{2}{\pi}\right)$ along with a new question: Find the average value of $f(x) = 3x^2 - 3$ on $[1, 3]$. A third card would have that answer (10) and so on. The answer to the last card goes on top of the first card.</p>
2	8.6	<p>Round Table</p> <p>In groups of four, each student has an identical paper with the same free-response question, along with four labeled boxes representing steps in the problem:</p> <ul style="list-style-type: none"> Identify all points of intersection. Set up the integral(s). Integrate by hand. Integrate using a calculator. <p>Have students complete the first step on their paper, and then pass the paper clockwise to another member in their group. That student checks the first step and then completes the second step on the paper. Students rotate again and the process continues until each student has their own paper back.</p>
3	8.9 8.10 8.11 8.12	<p>Quiz-Quiz-Trade</p> <p>Create cards with problems revolving around the x- or y-axis and others revolving around other axes (e.g., $y = x$ or $y = 3$). Give each student a card and have them write their answer on the back. Students quiz a partner about their own card then switch cards and repeat the process with a new partner.</p> <p>For the first round, concentrate on just setting up the integrals.</p> <p>In the second round, students can use their calculators to find the volume.</p>

TOPIC 8.1

Finding the Average Value of a Function on an Interval

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**CHA-4**

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE**CHA-4.B**

Determine the average value of a function using definite integrals.

ESSENTIAL KNOWLEDGE**CHA-4.B.1**The average value of a continuous function f over an interval $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$.

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Motion

TOPIC 8.2

Connecting Position, Velocity, and Acceleration of Functions Using Integrals

Required Course Content

ENDURING UNDERSTANDING

CHA-4

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE

CHA-4.C

Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion.

ESSENTIAL KNOWLEDGE

CHA-4.C.1

For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.

TOPIC 8.3

Using Accumulation Functions and Definite Integrals in Applied Contexts

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**CHA-4**

Definite integrals allow us to solve problems involving the accumulation of change over an interval.

LEARNING OBJECTIVE**CHA-4.D**

Interpret the meaning of a definite integral in accumulation problems.

CHA-4.E

Determine net change using definite integrals in applied contexts.

ESSENTIAL KNOWLEDGE**CHA-4.D.1**

A function defined as an integral represents an accumulation of a rate of change.

CHA-4.D.2

The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.

CHA-4.E.1

The definite integral can be used to express information about accumulation and net change in many applied contexts.

SUGGESTED SKILLS

Communication and
Notation

4.C

Use appropriate mathematical
symbols and notation.

TOPIC 8.4

Finding the Area Between Curves Expressed as Functions of x

Required Course Content

ENDURING UNDERSTANDING**CHA-5**

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE**CHA-5.A**Calculate areas in the plane
using the definite integral.**ESSENTIAL KNOWLEDGE****CHA-5.A.1**Areas of regions in the plane can be calculated
with definite integrals.

TOPIC 8.5

Finding the Area Between Curves Expressed as Functions of y

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING**CHA-5**

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE**CHA-5.A**

Calculate areas in the plane using the definite integral.

ESSENTIAL KNOWLEDGE**CHA-5.A.2**Areas of regions in the plane can be calculated using functions of either x or y .

SUGGESTED SKILLS

Connecting
Representations

2.B

Identify mathematical
information from graphical,
numerical, analytical, and/or
verbal representations.

TOPIC 8.6

Finding the Area Between Curves That Intersect at More Than Two Points

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.ACalculate areas in the plane
using the definite integral.

ESSENTIAL KNOWLEDGE

CHA-5.A.3Areas of certain regions in the plane may be
calculated using a sum of two or more definite
integrals or by evaluating a definite integral
of the absolute value of the difference of
two functions.

TOPIC 8.7

Volumes with Cross Sections: Squares and Rectangles

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**CHA-5**

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE**CHA-5.B**

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE**CHA-5.B.1**

Volumes of solids with square and rectangular cross sections can be found using definite integrals and the area formulas for these shapes.

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

ILLUSTRATIVE EXAMPLES

Illustrative examples of other cross sections in CHA-5.B.3:

- The volume of a funnel whose cross sections are circles can be found using the area formula for a circle and definite integrals.
- The volume of a solid whose cross sectional area is defined using a function can be found using the known area function and a definite integral.

TOPIC 8.8

Volumes with Cross Sections: Triangles and Semicircles

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.B

Calculate volumes of solids with known cross sections using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.B.2

Volumes of solids with triangular cross sections can be found using definite integrals and the area formulas for these shapes.

CHA-5.B.3

Volumes of solids with semicircular and other geometrically defined cross sections can be found using definite integrals and the area formulas for these shapes.

TOPIC 8.9

Volume with Disc Method: Revolving Around the x - or y -Axis

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**CHA-5**

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE**CHA-5.C**

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE**CHA-5.C.1**

Volumes of solids of revolution around the x - or y -axis may be found by using definite integrals with the disc method.

SUGGESTED SKILLS

Connecting
Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

TOPIC 8.10

Volume with Disc Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.2

Volumes of solids of revolution around any horizontal or vertical line in the plane may be found by using definite integrals with the disc method.

TOPIC 8.11

Volume with Washer Method: Revolving Around the x - or y -Axis

SUGGESTED SKILLS

Communication and Notation

4.E

Apply appropriate rounding procedures.

Required Course Content

ENDURING UNDERSTANDING**CHA-5**

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE**CHA-5.C**

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE**CHA-5.C.3**

Volumes of solids of revolution around the x - or y -axis whose cross sections are ring shaped may be found using definite integrals with the washer method.

SUGGESTED SKILLS

Connecting
Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCES

You can find the following related resource in AP Classroom:

- Classroom Resource > Volumes of Solids of Revolution

TOPIC 8.12

Volume with Washer Method: Revolving Around Other Axes

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.C

Calculate volumes of solids of revolution using definite integrals.

ESSENTIAL KNOWLEDGE

CHA-5.C.4

Volumes of solids of revolution around any horizontal or vertical line whose cross sections are ring shaped may be found using definite integrals with the washer method.

TOPIC 8.13

The Arc Length of a Smooth, Planar Curve and Distance Traveled **BC ONLY**

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**CHA-6**

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE**CHA-6.A**

Determine the length of a curve in the plane defined by a function, using a definite integral. **BC ONLY**

ESSENTIAL KNOWLEDGE**CHA-6.A.1**

The length of a planar curve defined by a function can be calculated using a definite integral. **BC ONLY**

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AP CALCULUS AB AND BC

UNIT **9** BC ONLY

Parametric Equations, Polar Coordinates, and Vector-Valued Functions



10–15% BC ONLY
AP EXAM WEIGHTING



~10–11
CLASS PERIODS



Remember to go to **AP Classroom** to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 9

Multiple-choice: ~25 questions

Free-response: 3 questions

Parametric Equations, Polar Coordinates, and Vector-Valued Functions



BIG IDEA 1

Change **CHA**

- How can we model motion not constrained to a linear path?

BIG IDEA 3

Analysis of Functions **FUN**

- How does the chain rule help us to analyze graphs defined using parametric equations or polar functions?

Developing Understanding

In this unit, students will build on their understanding of straight-line motion to solve problems in which particles are moving along curves in the plane. Students will define parametric equations and vector-valued functions to describe planar motion and apply calculus to solve motion problems. Students will learn that polar equations are a special case of parametric equations and will apply calculus to analyze graphs and determine lengths and areas. Have students treat this unit as an opportunity to reinforce past learning and transfer knowledge and skills to new situations, rather than as a new list of facts or strategies to memorize.

Building the Mathematical Practices

1.D 1.E 2.D 3.D

As students transition to parametric and vector-valued functions, they'll need to practice previously learned concepts and skills to reinforce the new procedures and representations they're learning in Unit 9. As with particle motion on a line, students learning to handle motion in the plane will need to practice interpreting which procedure is needed for different scenarios (differentiation or integration) and solving for speed, velocity, distance traveled, or initial position.

Reinforce the importance of precise notation, particularly regarding the variable of differentiation, as well as correct application of the chain rule. Leibniz notation helps students to remember how to find the derivative of y with respect to x for coordinates defined using the parameter t :

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}, \text{ provided } \frac{dx}{dt} \neq 0. \text{ Since } \frac{dy}{dx} \text{ is in terms of } t, \text{ students must be particularly careful when determining } \frac{d^2y}{dx^2}.$$

Similarly, using definite integrals to represent lengths and areas defined by polar curves is based on the same principles as calculating lengths and areas defined by the graphs of more familiar functions (i.e., the limit of a Riemann sum). Students will need to practice with trigonometric identities, radian measures and formulas for arc length and area of a sector to reinforce practice with associated calculus topics.

Preparing for the AP Exam

While students need more experience shifting mindsets from rectangular to polar coordinate systems, errors in arithmetic, algebra, trigonometry, and procedures such as the chain rule are often even more problematic. Provide opportunities for students to reinforce familiar skills and concepts as they practice new techniques in preparation for the AP Exam. As with analysis of graphs, sign charts can be useful tools for identifying answers to questions about the direction of motion or whether speed is increasing or decreasing, for example. To earn points for justification, however, students must connect their work to a relevant definition or theorem. Continue to emphasize accounting for initial values, as in past units, as well as precise communication and notational fluency. Paying attention to subscripts in problems involving more than one particle is essential to clear communication.

UNIT AT A GLANCE

Enduring Understanding	Topic	Suggested Skills	Class Periods
			~10–11 CLASS PERIODS
CHA-3	9.1 Defining and Differentiating Parametric Equations	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
	9.2 Second Derivatives of Parametric Equations	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
CHA-6	9.3 Finding Arc Lengths of Curves Given by Parametric Equations	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
CHA-3	9.4 Defining and Differentiating Vector-Valued Functions	1.D Identify an appropriate mathematical rule or procedure based on the relationship between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, anti-differentiation) to solve problems.	
FUN-8	9.5 Integrating Vector-Valued Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
	9.6 Solving Motion Problems Using Parametric and Vector-Valued Functions	1.E Apply appropriate mathematical rules or procedures, with and without technology.	
FUN-3	9.7 Defining Polar Coordinates and Differentiating in Polar Form	2.D Identify how mathematical characteristics or properties of functions are related in different representations.	
CHA-5	9.8 Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve	3.D Apply an appropriate mathematical definition, theorem, or test.	
	9.9 Finding the Area of the Region Bounded by Two Polar Curves	3.D Apply an appropriate mathematical definition, theorem, or test.	
 Go to AP Classroom to assign the Progress Check for Unit 9. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	9.2	Numbered Heads Together Organize the class into groups of four and give each student a number. Give the class one set of parametric equations and have students individually find the second derivative. When all students in a group have finished, have them stand up to discuss their answers, make any necessary corrections, and then sit back down. Choose a group and call a specific student number so that the student can share the answer with the class.
2	9.5 9.6	Scavenger Hunt Create cards containing a parametric initial value question and the answer to a different question. Students can start at any question, moving around the room to make a full circuit of all questions once complete. Focus on giving students an initial value and asking for one or both component values at a different t - or x -value (e.g., $x'(t) = 2t$, $y'(t) = \cos(t)$, and position is $(2, 0)$ at $t = \pi$. Find position at $t = \frac{5\pi}{6}$).
3	9.8	Create Representations Give students polar equations of various types: circle, limaçon with inner loop, cardioid, dimpled limaçon, rose curve, or lemniscates. Have students create a table of values, sketch the graph of the curve using rectangular coordinates, and sketch the graph of the curve using polar coordinates. Preface this activity by modeling the steps with one function on large paper, using wiki sticks to show the y -values as heights becoming the r -values as radii.
4	9.8	Paraphrasing Give students a proof which derives the polar area formula. Ask them to restate the meaning and derivation of this formula in their own words. Have them also compare and contrast this formula to the integration used to find areas under functions in the Cartesian coordinate system.
5	9.9	Stand Up, Hand Up, Pair Up Give each pair of students a polar area question to solve. Once they have completed both roles obtaining only the integral setup, have them use a calculator to find the numeric solution and confirm with you before standing up to switch pairs.

SUGGESTED SKILLS

Connecting
Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCES

- External Resource >
Davidson Next

You can find the following related resource in AP Classroom:

- Classroom Resource >
Vectors

TOPIC 9.1

Defining and Differentiating Parametric Equations

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to parametric functions. **BC ONLY**

CHA-3.G.2

For a curve defined parametrically, the value of $\frac{dy}{dx}$ at a point on the curve is the slope of

the line tangent to the curve at that point. $\frac{dy}{dx}$,

the slope of the line tangent to a curve defined using parametric equations, can be determined

by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$, provided $\frac{dx}{dt}$ does not

equal zero. **BC ONLY**

TOPIC 9.2

Second Derivatives of Parametric Equations

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.G

Calculate derivatives of parametric functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.G.3

$\frac{d^2 y}{dx^2}$ can be calculated by dividing $\frac{d}{dt} \left(\frac{dy}{dx} \right)$
by $\frac{dx}{dt}$. **BC ONLY**

AVAILABLE RESOURCES

- External Resource >
[Davidson Next](#)

You can find the following related resource in AP Classroom:

- Classroom Resource >
Vectors

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- External Resource >
Davidson Next

You can find the following related resource in AP Classroom:

- Classroom Resource >
Vectors

TOPIC 9.3

Finding Arc Lengths of Curves Given by Parametric Equations

Required Course Content

ENDURING UNDERSTANDING**CHA-6**

Definite integrals allow us to solve problems involving the accumulation of change in length over an interval.

LEARNING OBJECTIVE**CHA-6.B**

Determine the length of a curve in the plane defined by parametric functions, using a definite integral.

BC ONLY

ESSENTIAL KNOWLEDGE**CHA-6.B.1**

The length of a parametrically defined curve can be calculated using a definite integral.

BC ONLY

TOPIC 9.4

Defining and Differentiating Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

CHA-3

Derivatives allow us to solve real-world problems involving rates of change.

LEARNING OBJECTIVE

CHA-3.H

Calculate derivatives of vector-valued functions.

BC ONLY

ESSENTIAL KNOWLEDGE

CHA-3.H.1

Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions. **BC ONLY**

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.D

Identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.

AVAILABLE RESOURCES

- External Resource >
[Davidson Next](#)

You can find the following related resource in AP Classroom:

- Classroom Resource >
Vectors

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- External Resource >
Davidson Next

You can find the following related resource in AP Classroom:

- Classroom Resource >
Vectors

TOPIC 9.5

Integrating Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.A

Determine a particular solution given a rate vector and initial conditions.

BC ONLY

ESSENTIAL KNOWLEDGE

FUN-8.A.1

Methods for calculating integrals of real-valued functions can be extended to parametric or vector-valued functions. **BC ONLY**

TOPIC 9.6

Solving Motion Problems Using Parametric and Vector-Valued Functions

Required Course Content

ENDURING UNDERSTANDING

FUN-8

Solving an initial value problem allows us to determine an expression for the position of a particle moving in the plane.

LEARNING OBJECTIVE

FUN-8.B

Determine values for positions and rates of change in problems involving planar motion. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-8.B.1

Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along a curve in the plane defined using parametric or vector-valued functions.

BC ONLY

FUN-8.B.2

For a particle in planar motion over an interval of time, the definite integral of the velocity vector represents the particle's displacement (net change in position) over the interval of time, from which we might determine its position. The definite integral of speed represents the particle's total distance traveled over the interval of time. **BC ONLY**

SUGGESTED SKILLS

Implementing Mathematical Processes

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

You can find the following related resource in AP Classroom:

- Classroom Resource > Vectors

SUGGESTED SKILLS

Connecting
Representations

2.D

Identify how mathematical characteristics or properties of functions are related in different representations.

AVAILABLE RESOURCES

- External Resource >
[Davidson Next](#)

TOPIC 9.7

Defining Polar Coordinates and Differentiating in Polar Form

Required Course Content

ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

LEARNING OBJECTIVE

FUN-3.G

Calculate derivatives of functions written in polar coordinates. **BC ONLY**

ESSENTIAL KNOWLEDGE

FUN-3.G.1

Methods for calculating derivatives of real-valued functions can be extended to functions in polar coordinates. **BC ONLY**

FUN-3.G.2

For a curve given by a polar equation $r = f(\theta)$, derivatives of r , x , and y with respect to θ , and first and second derivatives of y with respect to x can provide information about the curve.

BC ONLY

TOPIC 9.8

Find the Area of a Polar Region or the Area Bounded by a Single Polar Curve

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.1

The concept of calculating areas in rectangular coordinates can be extended to polar coordinates. **BC ONLY**

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

TOPIC 9.9

Finding the Area of the Region Bounded by Two Polar Curves

Required Course Content

ENDURING UNDERSTANDING

CHA-5

Definite integrals allow us to solve problems involving the accumulation of change in area or volume over an interval.

LEARNING OBJECTIVE

CHA-5.D

Calculate areas of regions defined by polar curves using definite integrals. **BC ONLY**

ESSENTIAL KNOWLEDGE

CHA-5.D.2

Areas of regions bounded by polar curves can be calculated with definite integrals. **BC ONLY**

AP CALCULUS AB AND BC

UNIT **10** BC ONLY

Infinite Sequences and Series



15–20% BC ONLY
AP EXAM WEIGHTING



~17–18
CLASS PERIODS



Remember to go to **AP Classroom** to assign students the online **Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

Progress Check 10

Multiple-choice: ~45 questions

Free-response: 3 questions

Infinite Sequences and Series

BIG IDEA 1

Limits LIM

- How can the sum of infinitely many discrete terms be a finite value or represent a continuous function?



Developing Understanding

In this unit, students need to understand that a sum of infinitely many terms may converge to a finite value. They can develop intuition by exploring graphs, tables, and symbolic expressions for series that converge and diverge and for Taylor polynomials. Students should build connections to past learning, such as how evaluating improper integrals relates to the integral test or how using limiting cases of power series to represent continuous functions relates to differentiation and integration. Students who rely solely on memorizing a long list of tests and procedures typically find little success achieving a lasting conceptual understanding of series.

Building the Mathematical Practices

1.E 1.F 2.C 3.D

In Unit 10, students will need to develop proficiency with complex series notation and the ability to communicate their reasoning. Emphasize appropriate use of notation, precision of language, and establishing conditions for using a particular test. Remind students that a sound justification relies upon both mathematical evidence and reasons why that evidence supports the conclusion.

Additionally, students will need to practice determining which application is appropriate for different scenarios (for example, using the definitions of harmonic or p -series to classify certain infinite series) and then applying associated procedures accurately. Students will also need to practice using Taylor polynomials to approximate the value of a function, choosing and implementing an appropriate method to bound the error involved in the approximation, and effectively communicating supporting work.

Connecting representations is an important skill to develop in this unit. For example, students will need to identify infinite power series to represent functions presented symbolically or move between graphic and symbolic representations of an interval of convergence.

Preparing for the AP Exam

Students are more likely to demonstrate an incomplete understanding of series or to struggle with communicating their understanding of it compared to other topics. Continue to model and expect correct notation and language to present solutions, explain reasoning, and justify conclusions. For example, using the ratio test to find a radius of convergence, or operating on a known series to create another series, requires proficient, well-presented algebra. Applying a convergence test requires explicit verification that all necessary conditions are met. Determining that a given number is an error bound requires calculating an appropriate value and communicating that the value is less than the given number.

Intentional focus on the recurrent theme of using limiting cases to move from discrete approximations to analytic calculations and determinations is one way to help students to finish the year with a strong performance on the AP Exam and to come away with an enduring, meaningful understanding of calculus.

UNIT AT A GLANCE

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~17–18 CLASS PERIODS
LIM-7	10.1 Defining Convergent and Divergent Infinite Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.2 Working with Geometric Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.3 The n th Term Test for Divergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.4 Integral Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.5 Harmonic Series and p -Series	3.B Identify an appropriate mathematical definition, theorem, or test to apply.	
	10.6 Comparison Tests for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.7 Alternating Series Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.8 Ratio Test for Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.9 Determining Absolute or Conditional Convergence	3.D Apply an appropriate mathematical definition, theorem, or test.	
	10.10 Alternating Series Error Bound	1.E Apply appropriate mathematical rules or procedures, with and without technology.	

(Continued)

UNIT AT A GLANCE (cont'd)

Enduring Understanding			Class Periods
	Topic	Suggested Skills	~17–18 CLASS PERIODS
LIM-8	10.11 Finding Taylor Polynomial Approximations of Functions	3.D Apply an appropriate mathematical definition, theorem, or test. 2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.12 Lagrange Error Bound	1.F Explain how an approximated value relates to the actual value.	
	10.13 Radius and Interval of Convergence of Power Series	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.14 Finding Taylor or Maclaurin Series for a Function	2.C Identify a re-expression of mathematical information presented in a given representation.	
	10.15 Representing Functions as Power Series	3.D Apply an appropriate mathematical definition, theorem, or test.	
 Go to AP Classroom to assign the Progress Check for Unit 10. Review the results in class to identify and address any student misunderstandings.			

SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional. They were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit.

Activity	Topic	Sample Activity
1	10.1	<p>Predict and Confirm</p> <p>Demonstrate a geometric series, the harmonic series, and the alternating series by distributing pieces of a donut, pizza, or licorice. Ask the class to predict how much the student(s) will eventually receive in total. For instance, give three students each one-fourth, then one-fourth of the remaining fourth, and so on. Students should guess that since the remaining part is approaching zero, each student will eventually receive one-third.</p> <p>For alternating harmonic series, give one student a whole piece, then take away $\frac{1}{2}$, then give $\frac{1}{3}$, then take away $\frac{1}{4}$, and so forth.</p>
2	10.2 10.3 10.4 10.5 10.6 10.7 10.8	<p>Graphic Organizer</p> <p>Put students in groups with poster paper and have them organize and explain all the series tests using pictures, text, flowcharts, cartoons, or other drawings. Have them include each test's conditions and how to choose which test to apply.</p>
3	10.13	<p>Odd One Out</p> <p>Begin by modeling an example, such as three images that are pieces of furniture and one image of a houseplant, explaining that the houseplant is the "odd one out" because it's not like the other images. Then distribute a set of series such that all of the series, except one, have something in common. For example, all of the series except one could have the same type of interval of convergence (open, closed, open at left or right endpoint). Or they could converge only at the series' center or they could converge for all real numbers. Then have students decide which series in the set is the "odd one out."</p>
4	10.15	<p>Collaborative Poster</p> <p>Use this strategy for groups to create their own free-response questions. Give each group of four students a basic series, such as $\sin(x)$, $\cos(x)$, e^x, or $\frac{1}{1+x}$. Ask the first two members to choose a manipulation of the series and show work to complete the task. Ask the final two members to watch silently and confirm the first two steps. Then the final two members choose further actions to perform on the new series (i.e., differentiate, integrate, and find interval of convergence).</p>

TOPIC 10.1

Defining Convergent and Divergent Infinite Series

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**LIM-7**

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE**LIM-7.A**

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE**LIM-7.A.1**

The n th partial sum is defined as the sum of the first n terms of a series. **BC ONLY**

LIM-7.A.2

An infinite series of numbers converges to a real number S (or has sum S), if and only if the limit of its sequence of partial sums exists and equals S . **BC ONLY**

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

TOPIC 10.2

Working with Geometric Series

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.3

A geometric series is a series with a constant ratio between successive terms. **BC ONLY**

LIM-7.A.4

If a is a real number and r is a real number such that $|r| < 1$, then the geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}. \quad \text{BC ONLY}$$

TOPIC 10.3

The n th Term Test for Divergence

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**LIM-7**

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE**LIM-7.A**

Determine whether a series converges or diverges.

BC ONLY**ESSENTIAL KNOWLEDGE****LIM-7.A.5**

The n th term test is a test for divergence of a series. **BC ONLY**

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

TOPIC 10.4

Integral Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.6

The integral test is a method to determine whether a series converges or diverges.

BC ONLY

TOPIC 10.5

Harmonic Series and p -Series

SUGGESTED SKILLS

Justification

3.B

Identify an appropriate mathematical definition, theorem, or test to apply.

Required Course Content

ENDURING UNDERSTANDING**LIM-7**

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE**LIM-7.A**

Determine whether a series converges or diverges.

BC ONLY**ESSENTIAL KNOWLEDGE****LIM-7.A.7**

In addition to geometric series, common series of numbers include the harmonic series, the alternating harmonic series, and p -series.

BC ONLY**AVAILABLE RESOURCES**

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

TOPIC 10.6

Comparison Tests for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.8

The comparison test is a method to determine whether a series converges or diverges.

BC ONLY

LIM-7.A.9

The limit comparison test is a method to determine whether a series converges or diverges. **BC ONLY**

TOPIC 10.7

Alternating Series Test for Convergence

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Required Course Content

ENDURING UNDERSTANDING**LIM-7**

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE**LIM-7.A**

Determine whether a series converges or diverges.

BC ONLY**ESSENTIAL KNOWLEDGE****LIM-7.A.10**

The alternating series test is a method to determine whether an alternating series converges. **BC ONLY**

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

TOPIC 10.8

Ratio Test for Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.11

The ratio test is a method to determine whether a series of numbers converges or diverges.

BC ONLY**X** EXCLUSION STATEMENT

The n th term test for divergence, and the integral test, comparison test, limit comparison test, alternating series test, and ratio test for convergence are assessed on the AP Calculus BC Exam. Other methods are not assessed on the exam. However, teachers may include additional methods in the course, if time permits.

TOPIC 10.9

Determining Absolute or Conditional Convergence

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.A

Determine whether a series converges or diverges.

BC ONLY

ESSENTIAL KNOWLEDGE

LIM-7.A.12

A series may be absolutely convergent, conditionally convergent, or divergent.

BC ONLY

LIM-7.A.13

If a series converges absolutely, then it converges. **BC ONLY**

LIM-7.A.14

If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value. **BC ONLY**

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.E

Apply appropriate
mathematical rules or
procedures, with and without
technology.

AVAILABLE RESOURCES

- External Resource >
[Davidson Next](#)

TOPIC 10.10

Alternating Series Error Bound

Required Course Content

ENDURING UNDERSTANDING

LIM-7

Applying limits may allow us to determine the finite sum of infinitely many terms.

LEARNING OBJECTIVE

LIM-7.B

Approximate the sum of a
series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-7.B.1

If an alternating series converges by the
alternating series test, then the alternating
series error bound can be used to bound how
far a partial sum is from the value of the infinite
series. **BC ONLY**

TOPIC 10.11

Finding Taylor Polynomial Approximations of Functions

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.A

Represent a function at a point as a Taylor polynomial.

BC ONLY

LIM-8.B

Approximate function values using a Taylor polynomial. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.A.1

The coefficient of the n th degree term in a Taylor polynomial for a function f centered at

$$x = a \text{ is } \frac{f^{(n)}(a)}{n!}. \text{ BC ONLY}$$

LIM-8.A.2

In many cases, as the degree of a Taylor polynomial increases, the n th degree polynomial will approach the original function over some interval. **BC ONLY**

LIM-8.B.1

Taylor polynomials for a function f centered at $x = a$ can be used to approximate function values of f near $x = a$. **BC ONLY**

SUGGESTED SKILLS

Justification

3.D

Apply an appropriate mathematical definition, theorem, or test.

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Implementing
Mathematical Processes

1.F

Explain how an approximated
value relates to the actual value.

AVAILABLE RESOURCES

- External Resource >
Davidson Next

You can find the following
related resource in AP
Classroom:

- Classroom Resource >
Approximation

TOPIC 10.12

Lagrange Error Bound

Required Course Content

ENDURING UNDERSTANDING**LIM-8**

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE**LIM-8.C**

Determine the error bound
associated with a Taylor
polynomial approximation. **BC
ONLY**

ESSENTIAL KNOWLEDGE**LIM-8.C.1**

The Lagrange error bound can be used to
determine a maximum interval for the error of a
Taylor polynomial approximation to a function.
BC ONLY

LIM-8.C.2

In some situations, the alternating series error
bound can be used to bound the error of a
Taylor polynomial approximation to the value of
a function. **BC ONLY**

TOPIC 10.13

Radius and Interval of Convergence of Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.D

Determine the radius of convergence and interval of convergence for a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.D.1

A power series is a series of the form $\sum_{n=0}^{\infty} a_n(x-r)^n$, where n is a non-negative integer, $\{a_n\}$ is a sequence of real numbers, and r is a real number. **BC ONLY**

LIM-8.D.2

If a power series converges, it either converges at a single point or has an interval of convergence. **BC ONLY**

LIM-8.D.3

The ratio test can be used to determine the radius of convergence of a power series. **BC ONLY**

LIM-8.D.4

The radius of convergence of a power series can be used to identify an open interval on which the series converges, but it is necessary to test both endpoints of the interval to determine the interval of convergence. **BC ONLY**

LIM-8.D.5

If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval. **BC ONLY**

LIM-8.D.6

The radius of convergence of a power series obtained by term-by-term differentiation or term-by-term integration is the same as the radius of convergence of the original power series. **BC ONLY**

SUGGESTED SKILLS

Connecting Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

SUGGESTED SKILLS

Connecting
Representations

2.C

Identify a re-expression of mathematical information presented in a given representation.

AVAILABLE RESOURCES

- External Resource >
[Davidson Next](#)

TOPIC 10.14

Finding Taylor or Maclaurin Series for a Function

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.E

Represent a function as a Taylor series or a Maclaurin series. **BC ONLY**

LIM-8.F

Interpret Taylor series and Maclaurin series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.E.1

A Taylor polynomial for $f(x)$ is a partial sum of the Taylor series for $f(x)$. **BC ONLY**

LIM-8.F.1

The Maclaurin series for $\frac{1}{1-x}$ is a geometric series. **BC ONLY**

LIM-8.F.2

The Maclaurin series for $\sin x$, $\cos x$, and e^x provides the foundation for constructing the Maclaurin series for other functions. **BC ONLY**

TOPIC 10.15

Representing Functions as Power Series

Required Course Content

ENDURING UNDERSTANDING

LIM-8

Power series allow us to represent associated functions on an appropriate interval.

LEARNING OBJECTIVE

LIM-8.G

Represent a given function as a power series. **BC ONLY**

ESSENTIAL KNOWLEDGE

LIM-8.G.1

Using a known series, a power series for a given function can be derived using operations such as term-by-term differentiation or term-by-term integration, and by various methods (e.g., algebraic processes, substitutions, or using properties of geometric series). **BC ONLY**

SUGGESTED SKILLS

Justification

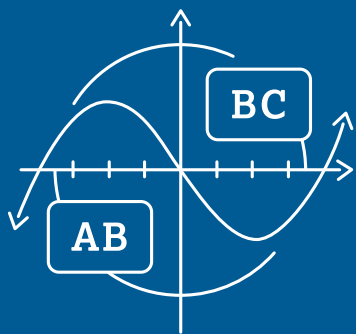
3.D

Apply an appropriate mathematical definition, theorem, or test.

AVAILABLE RESOURCES

- External Resource > [Davidson Next](#)

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AP CALCULUS AB AND BC

Exam Information

Exam Overview

The AP Calculus AB and BC Exams assess student understanding of the mathematical practices and learning objectives outlined in the course framework. The exams are both 3 hours and 10 minutes long and include 42 multiple-choice questions and 6 free-response questions. The details of the exams, including exam weighting, timing, and calculator requirements, can be found below:

Section	Question Type	Number of Questions	Exam Weighting	Timing
I	Multiple-choice questions			
	Part A: Graphing calculator not permitted	29	35%	62 minutes
	Part B: Graphing calculator required	13	15%	38 minutes
II	Free-response questions			
	Part A: Graphing calculator required	2	16.7%	30 minutes
	Part B: Graphing calculator not permitted	4	33.3%	60 minutes

Accessible technology that has the capabilities expected for AP Calculus is available for students who are blind or visually impaired. This technology should be used during the course, and an accommodation request to use this technology on the AP Exam must be made through the College Board's [Services for Students with Disabilities \(SSD\)](#).

The exams assess content from the three big ideas of the course.

Big Idea 1: Change

Big Idea 2: Limits

Big Idea 3: Analysis of Functions

The AP Exams also assess each of the units of the course—eight units for AP Calculus AB and 10 for AP Calculus BC—with the following exam weighting on the multiple-choice section:

Exam Weighting for the Multiple-Choice Section of the AP Exam

Unit	Exam Weighting	
	AB	BC
Unit 1: Limits and Continuity	10–15%	5–10%
Unit 2: Differentiation: Definition and Basic Derivative Rules	10–15%	5–10%
Unit 3: Differentiation: Composite, Implicit, and Inverse Functions	5–10%	5–10%
Unit 4: Contextual Applications of Differentiation	10–15%	5–10%
Unit 5: Applying Derivatives to Analyze Functions	15–20%	10–15%
Unit 6: Integration and Accumulation of Change	15–20%	15–20%
Unit 7: Differential Equations	5–10%	5–10%
Unit 8: Applications of Integration	10–15%	5–10%
Unit 9: Parametric Equations, Polar Coordinates, and Vector-Valued Functions BC ONLY		10–15%
Unit 10: Infinite Sequences and Series BC ONLY		15–20%

How Student Learning Is Assessed on the AP Exam

Section I: Multiple-Choice

The first section of the AP Calculus AB and BC Exams includes 42 multiple-choice questions. A graphing calculator is required for the final 13 questions (Part B). Refer to the Technology Needs and Use of Graphing Calculators section on p. 7 for a list of calculator capabilities necessary for the AP Exam. Both the AB and BC Exams include algebraic, exponential, logarithmic, trigonometric, and general types of functions. Both also include analytical, graphical, tabular, and verbal types of representations.

Mathematical Practices 1, 2, and 3 are assessed in the multiple-choice section, with the following exam weighting (Practice 4 is not assessed):

Exam Weighting for the Multiple-Choice Section of the AP Exam

Mathematical Practice	Exam Weighting
Practice 1: Implementing Mathematical Processes	50–70%
Practice 2: Connecting Representations	15–30%
Practice 3: Justification	10–20%

Section II: Free-Response

The second section of the AP Calculus AB and BC Exams includes six free-response questions with a variety of content topics from across the units of the course. The AB and BC Exams include three common free-response questions that assess content from the domain of the AB Calculus course. Both AP Exams include various types of functions and function representations and a roughly equal mix of procedural and conceptual tasks. They both also include at least two questions that incorporate a real-world context or scenario into the question.

All four mathematical practices are assessed in the free-response section with the following exam weighting:

Exam Weighting for the Free-Response Section of the AP Exam

Mathematical Practice	Exam Weighting
Practice 1: Implementing Mathematical Processes	35–60%
Practice 2: Connecting Representations	10–20%
Practice 3: Justification	35–60%
Practice 4: Communication and Notation	10–25%

Task Verbs Used in Free-Response Questions

The following task verbs are commonly used in the free-response questions:

- **Approximate:** Use rounded decimal values or other estimates in calculations, which require writing an expression to show work.
- **Calculate/Write an expression:** Write an appropriate expression or equation to answer a question. Unless otherwise directed, calculations also require evaluating an expression or solving an equation, but the expression or equation must also be presented to show work. "Calculate" tasks might also be formulated as "How many?" or "What is the value?"
- **Determine:** Apply an appropriate definition, theorem, or test to identify values, intervals, or solutions whose existence or uniqueness can be established. "Determine" tasks may also be phrased as "Find."
- **Estimate:** Use models or representations to find approximate values for functions.
- **Evaluate:** Apply mathematical processes, including the use of appropriate rounding procedures, to find the value of an expression at a given point or over a given interval.
- **Explain:** Use appropriate definitions or theorems to provide reasons or rationales for solutions and conclusions. "Explain" tasks may also be phrased as "Give a reason for..."
- **Identify/Indicate:** Indicate or provide information about a specified topic, without elaboration or explanation.
- **Interpret:** Describe the connection between a mathematical expression or solution and its meaning within the realistic context of a problem, often including consideration of units.
- **Interpret (when given a representation):** Identify mathematical information represented graphically, numerically, analytically, and/or verbally, with and without technology.
- **Justify:** Identify a logical sequence of mathematical definitions, theorems, or tests to support an argument or conclusion, explain why these apply, and then apply them.
- **Represent:** Use appropriate graphs, symbols, words, and/or tables of numerical values to describe mathematical concepts, characteristics, and/or relationships.
- **Verify:** Confirm that the conditions of a mathematical definition, theorem, or test are met in order to explain why it applies in a given situation. Alternately, confirm that solutions are accurate and appropriate.

Sample AP Calculus AB and BC Exam Questions

The sample exam questions that follow illustrate the relationship between the course framework and the AP Calculus AB and BC Exams and serve as examples of the types of questions that appear on the exams. After the sample questions is a table that shows which skill, learning objective(s), and unit each question relates to. The table also provides the answers to the multiple-choice questions.

Section I: Multiple-Choice

PART A (AB or BC)

Graphing calculators are not permitted on this part of the exam.

1. $\lim_{x \rightarrow 0} \frac{1 - \cos^2(2x)}{(2x)^2} =$
- (A) 0
- (B) $\frac{1}{4}$
- (C) $\frac{1}{2}$
- (D) 1

$$f(x) = \begin{cases} \frac{2}{x} & \text{for } x < -1 \\ x^2 - 3 & \text{for } -1 \leq x \leq 2 \\ 4x - 3 & \text{for } x > 2 \end{cases}$$

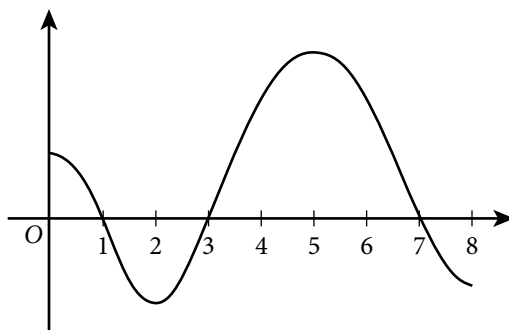
2. Let f be the function defined above. At what values of x , if any, is f not differentiable?
- (A) $x = -1$ only
- (B) $x = 2$ only
- (C) $x = -1$ and $x = -2$
- (D) f is differentiable for all values of x .

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	-4	-5	3
2	-3	1	8	4

3. The table above gives values of the differentiable functions f and g and their derivatives at selected values of x . If h is the function defined by $h(x) = f(x)g(x) + 2g(x)$, then $h'(1) =$
- (A) 32
(B) 30
(C) -6
(D) -16
4. If $x^3 - 2xy + 3y^2 = 7$, then $\frac{dy}{dx} =$
- (A) $\frac{3x^2 + 4y}{2x}$
(B) $\frac{3x^2 - 2y}{2x - 6y}$
(C) $\frac{3x^2}{2x - 6y}$
(D) $\frac{3x^2}{2 - 6y}$
5. The radius of a right circular cylinder is increasing at a rate of 2 units per second. The height of the cylinder is decreasing at a rate of 5 units per second. Which of the following expressions gives the rate at which the volume of the cylinder is changing with respect to time in terms of the radius r and height h of the cylinder?
(The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
- (A) $-20\pi r$
(B) $-2\pi rh$
(C) $4\pi rh - 5\pi r^2$
(D) $4\pi rh + 5\pi r^2$

6. Which of the following is equivalent to the definite integral $\int_2^6 \sqrt{x} \, dx$?

- (A) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{4}{n} \sqrt{\frac{4k}{n}}$
 (B) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{6}{n} \sqrt{\frac{6k}{n}}$
 (C) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{4}{n} \sqrt{2 + \frac{4k}{n}}$
 (D) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{6}{n} \sqrt{2 + \frac{6k}{n}}$



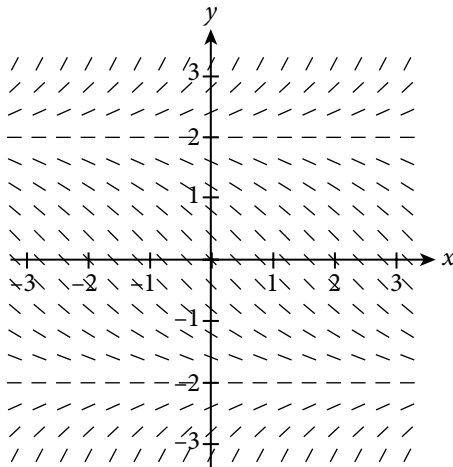
Graph of g

7. The figure above shows the graph of the continuous function g on the interval $[0, 8]$. Let h be the function defined by $h(x) = \int_3^x g(t) \, dt$. On what intervals is h increasing?

- (A) $[2, 5]$ only
 (B) $[1, 7]$
 (C) $[0, 1]$ and $[3, 7]$
 (D) $[1, 3]$ and $[7, 8]$

8. $\int \frac{x}{\sqrt{1-9x^2}} \, dx =$

- (A) $-\frac{1}{9} \sqrt{1-9x^2} + C$
 (B) $-\frac{1}{18} \ln \sqrt{1-9x^2} + C$
 (C) $\frac{1}{3} \arcsin(3x) + C$
 (D) $\frac{x}{3} \arcsin(3x) + C$



9. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = \frac{y-2}{2}$

(B) $\frac{dy}{dx} = \frac{y^2-4}{4}$

(C) $\frac{dy}{dx} = \frac{x-2}{2}$

(D) $\frac{dy}{dx} = \frac{x^2-4}{4}$

10. Let R be the region bounded by the graph of $x = e^y$, the vertical line $x = 10$, and the horizontal lines $y = 1$ and $y = 2$. Which of the following gives the area of R ?

(A) $\int_1^2 e^y dy$

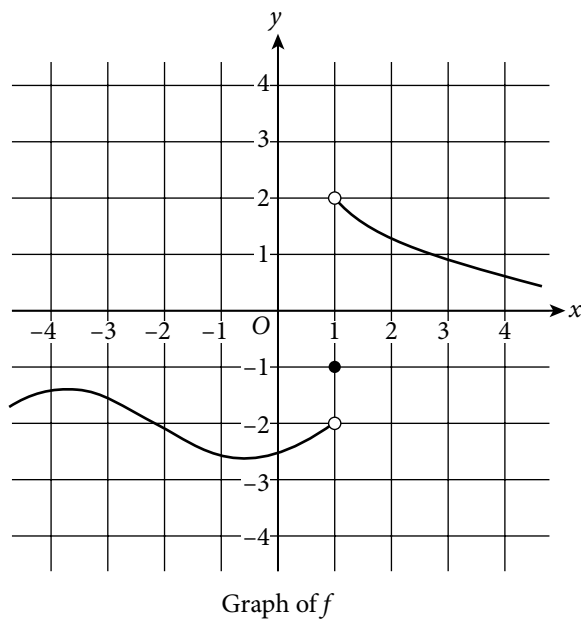
(B) $\int_e^{e^2} \ln x dx$

(C) $\int_1^2 (10 - e^y) dy$

(D) $\int_e^{10} (\ln x - 1) dx$

PART B (AB or BC)

A graphing calculator is required on this part of the exam.



11. The graph of the function f is shown in the figure above. The value of $\lim_{x \rightarrow 1^+} f(x)$ is
- (A) -2
(B) -1
(C) 2
(D) nonexistent
12. The velocity of a particle moving along a straight line is given by $v(t) = 1.3t \ln(0.2t + 0.4)$ for time $t \geq 0$. What is the acceleration of the particle at time $t = 1.2$?
- (A) -0.580
(B) -0.548
(C) -0.093
(D) 0.660

x	-1	0	2	4	5
$f'(x)$	11	9	8	5	2

13. Let f be a twice-differentiable function. Values of f' , the derivative of f , at selected values of x are given in the table above. Which of the following statements must be true?
- (A) f is increasing for $-1 \leq x \leq 5$.
- (B) The graph of f is concave down for $-1 < x < 5$.
- (C) There exists c , where $-1 < c < 5$, such that $f'(c) = -\frac{3}{2}$.
- (D) There exists c , where $-1 < c < 5$, such that $f''(c) = -\frac{3}{2}$.
14. Let f be the function with derivative defined by $f'(x) = 2 + (2x - 8)\sin(x + 3)$. How many points of inflection does the graph of f have on the interval $0 < x < 9$?
- (A) One
- (B) Two
- (C) Three
- (D) Four
15. Honey is poured through a funnel at a rate of $r(t) = 4e^{-0.35t}$ ounces per minute, where t is measured in minutes. How many ounces of honey are poured through the funnel from time $t = 0$ to time $t = 3$?
- (A) 0.910
- (B) 1.400
- (C) 2.600
- (D) 7.429

PART A (BC only)

Graphing calculators are not permitted on this part of the exam.

x	2	5
$f(x)$	4	7
$f'(x)$	2	3

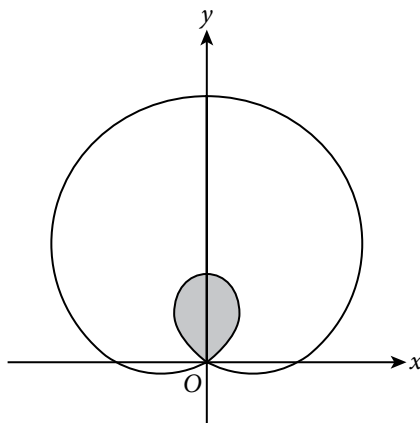
16. The table above gives values of the differentiable function f and its derivative f' at selected values of x .
If $\int_2^5 f(x)dx = 14$, what is the value of $\int_2^5 x \cdot f'(x)dx$?
- (A) 13
(B) 27
(C) $\frac{63}{2}$
(D) 41
17. The number of fish in a lake is modeled by the function F that satisfies the logistic differential equation $\frac{dF}{dt} = 0.04F \left(1 - \frac{F}{5000} \right)$, where t is the time in months and $F(0) = 2000$. What is $\lim_{t \rightarrow \infty} F(t)$?
- (A) 10,000
(B) 5000
(C) 2500
(D) 2000
18. A curve is defined by the parametric equations $x(t) = t^2 + 3$ and $y(t) = \sin(t^2)$. Which of the following is an expression for $\frac{d^2y}{dx^2}$ in terms of t ?
- (A) $-\sin(t^2)$
(B) $-2t \sin(t^2)$
(C) $\cos(t^2) - 2t^2 \sin(t^2)$
(D) $2\cos(t^2) - 4t^2 \sin(t^2)$
19. Which of the following series is conditionally convergent?
- (A) $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^3 + 1}$
(B) $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k + 1}$
(C) $\sum_{k=1}^{\infty} (-1)^k \frac{5k}{k + 1}$
(D) $\sum_{k=1}^{\infty} (-1)^k \frac{5k^2}{k + 1}$

20. Let f be the function defined by $f(x) = e^{2x}$. Which of the following is the Maclaurin series for f' , the derivative of f ?

- (A) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
- (B) $2 + 2x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \cdots + \frac{2x^n}{n!} + \cdots$
- (C) $1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots + \frac{(2x)^n}{n!} + \cdots$
- (D) $2 + 2(2x) + \frac{2(2x)^2}{2!} + \frac{2(2x)^3}{3!} + \cdots + \frac{2(2x)^n}{n!} + \cdots$

PART B (BC only)

A graphing calculator is required on this part of the exam.



21. The figure above shows the graph of the polar curve $r = 2 + 4\sin \theta$. What is the area of the shaded region?
- (A) 2.174
(B) 2.739
(C) 13.660
(D) 37.699
22. The function f has derivatives of all orders for all real numbers. It is known that $|f^{(4)}(x)| \leq \frac{12}{5}$ and $|f^{(5)}(x)| \leq \frac{3}{2}$ for $0 \leq x \leq 2$. Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. The Taylor series for f about $x = 0$ converges at $x = 2$. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(2) - P_4(2)| \leq k$?
- (A) $\frac{2^5}{5!} \cdot \frac{3}{2}$
(B) $\frac{2^5}{5!} \cdot \frac{12}{5}$
(C) $\frac{2^4}{4!} \cdot \frac{3}{2}$
(D) $\frac{2^4}{4!} \cdot \frac{12}{5}$

Section II: Free-Response

The following are examples of the kinds of free-response questions found on the exam.

PART A (AB or BC)

A graphing calculator is required on this part of the exam.

QUESTION 1

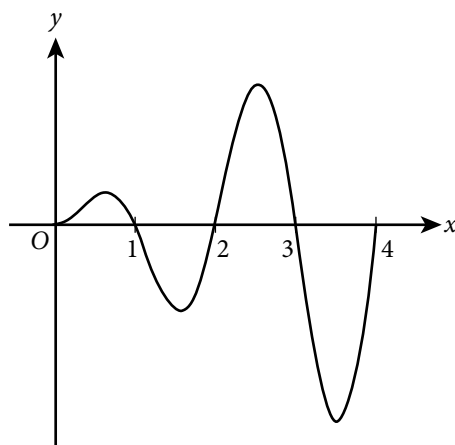
t (hours)	0	2	4	6	8	10	12
$R(t)$ (vehicles per hour)	2935	3653	3442	3010	3604	1986	2201

On a certain weekday, the rate at which vehicles cross a bridge is modeled by the differentiable function R for $0 \leq t \leq 12$, where $R(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 A.M. ($t = 0$). Values of $R(t)$ for selected values of t are given in the table above.

- Use the data in the table to approximate $R'(5)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $R'(5)$ in the context of the problem.
- Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\int_0^{12} R(t) dt$. Indicate units of measure.
- On a certain weekend day, the rate at which vehicles cross the bridge is modeled by the function H defined by $H(t) = -t^3 - 3t^2 + 288t + 1300$ for $0 \leq t \leq 17$, where $H(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 A.M. ($t = 0$). According to this model, what is the average number of vehicles crossing the bridge per hour on the weekend day for $0 \leq t \leq 12$?
- For $12 < t < 17$, $L(t)$, the local linear approximation to the function H given in part C at $t = 12$, is a better model for the rate at which vehicles cross the bridge on the weekend day. Use $L(t)$ to find the time t , for $12 < t < 17$, at which the rate of vehicles crossing the bridge is 2000 vehicles per hour. Show the work that leads to your answer.

PART B (AB or BC)

Graphing calculators are not permitted on this part of the exam.

QUESTION 2

Graph of f'

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the closed interval $[0, 4]$. The areas of the regions bounded by the graph of f' and the x -axis on the intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ are 2, 6, 10, and 14, respectively. The graph of f' has horizontal tangents at $x = 0.6$, $x = 1.6$, $x = 2.5$, and $x = 3.5$. It is known that $f(2) = 5$.

- A. On what open intervals contained in $(0, 4)$ is the graph of f both decreasing and concave down? Give a reason for your answer.
- B. Find the absolute minimum value of f on the interval $[0, 4]$. Justify your answer.
- C. Evaluate $\int_0^4 f(x)f'(x)dx$.
- D. The function g is defined by $g(x) = x^3 f(x)$. Find $g'(2)$. Show the work that leads to your answer.

PART A (BC only)

A graphing calculator is required on this part of the exam.

QUESTION 3

For $0 \leq t \leq 5$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, -7)$. It is known that

$$\frac{dx}{dt} = \sin\left(\frac{t}{t+3}\right) \text{ and } \frac{dy}{dt} = e^{\cos t}.$$

- A. Write an equation for the line tangent to the curve at the point $(2, -7)$.
- B. Find the y -coordinate of the position of the particle at time $t = 4$.
- C. Find the total distance traveled by the particle from time $t = 1$ to time $t = 4$.
- D. Find the time at which the speed of the particle is 2.5. Find the acceleration vector of the particle at this time.

PART B (BC only)

Graphing calculators are not permitted on this part of the exam.

QUESTION 4

The Maclaurin series for the function f is given by

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k^2} = x - \frac{x^2}{4} + \frac{x^3}{9} - \cdots \text{ on its interval of convergence.}$$

- A. Use the ratio test to determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.
- B. The Maclaurin series for f evaluated at $x = \frac{1}{4}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $f\left(\frac{1}{4}\right)$ using the first two nonzero terms of this series is $\frac{15}{64}$. Show that this approximation differs from $f\left(\frac{1}{4}\right)$ by less than $\frac{1}{500}$.
- C. Let h be the function defined by $h(x) = \int_0^x f(t) dt$. Write the first three nonzero terms and the general term of the Maclaurin series for h .

Answer Key and Question Alignment to Course Framework

Multiple-Choice Question	Answer	Skill	Learning Objective	Unit
1	D	1.E	LIM-1.E	1
2	B	3.D	FUN-2.A	2
3	A	1.E	FUN-3.B	2
4	B	1.E	FUN-3.D	3
5	C	1.E	CHA-3.D	4
6	C	2.C	LIM-5.C	6
7	C	2.D	FUN-5.A	6
8	A	1.E	FUN-6.D	6
9	B	2.C	FUN-7.C	7
10	C	1.D	CHA-5.A	8
11	C	2.B	LIM-1.C	1
12	C	1.E	CHA-3.B	4
13	D	3.D	FUN-1.B	5
14	D	2.D	FUN-4.A	5
15	D	3.D	CHA-4.D	8
16	A	1.E	FUN-6.E	6
17	B	3.F	FUN-7.H	7
18	A	1.E	CHA-3.G	9
19	B	3.D	LIM-7.A	10
20	D	3.D	LIM-8.G	10
21	A	3.D	CHA-5.D	9
22	A	1.F	LIM-8.C	10

Free-Response Question	Skills	Learning Objective	Unit
1	1.D, 1.E, 2.B, 3.F	CHA-2.D, CHA-3.A, CHA-3.C, CHA-3.F, CHA-4.B, LIM-5.A	2, 4, 6, 8
2	1.C, 1.E, 2.B, 2.E, 3.B, 3.E	FUN-3.B, FUN-4.A, FUN-5.A, FUN-6.D	2, 5, 6
3	1.C, 1.D, 1.E, 2.B	CHA-3.G, FUN-8.B	9
4	1.D, 1.E, 3.B, 3.D, 3.E	LIM-7.A, LIM-7.B, LIM-8.D, LIM-8.G	10

The scoring information for the questions within this course and exam description, along with further exam resources, can be found on the [AP Calculus AB Exam Page](#) and the [AP Calculus BC Exam Page](#) on AP Central.

Scoring Guidelines

Scoring Guidelines for Question 1

Part A (AB or BC): Graphing Calculator Required

9 points

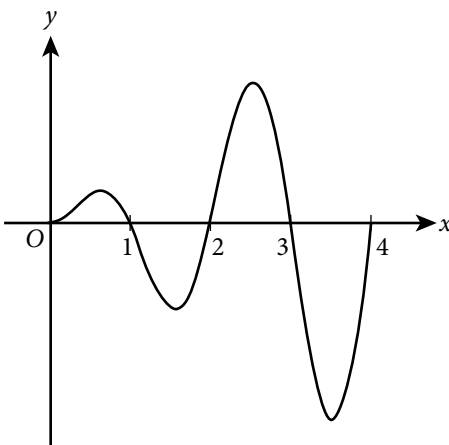
t (hours)	0	2	4	6	8	10	12
$R(t)$ (vehicles per hour)	2935	3653	3442	3010	3604	1986	2201

On a certain weekday, the rate at which vehicles cross a bridge is modeled by the differentiable function R for $0 \leq t \leq 12$, where $R(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 A.M. ($t = 0$).

Values of $R(t)$ for selected values of t are given in the table above.

- Use the data in the table to approximate $R'(5)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $R'(5)$ in the context of the problem.
- Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\int_0^{12} R(t) dt$. Indicate units of measure.
- On a certain weekend day, the rate at which vehicles cross the bridge is modeled by the function H defined by $H(t) = -t^3 - 3t^2 + 288t + 1300$ for $0 \leq t \leq 17$, where $H(t)$ is measured in vehicles per hour and t is the number of hours since 7:00 A.M. ($t = 0$). According to this model, what is the average number of vehicles crossing the bridge per hour on the weekend day for $0 \leq t \leq 12$?
- For $12 < t < 17$, $L(t)$, the local linear approximation to the function H given in part C at $t = 12$, is a better model for the rate at which vehicles cross the bridge on the weekend day. Use $L(t)$ to find the time t , for $12 < t < 17$, at which the rate of vehicles crossing the bridge is 2000 vehicles per hour. Show the work that leads to your answer.

Model Solution	Scoring
<p>A. Use the data in the table to approximate $R'(5)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $R'(5)$ in the context of the problem.</p> $R'(5) \approx \frac{R(6) - R(4)}{6 - 4} = \frac{3010 - 3442}{2} = -216$ <p>At time $t = 5$ hours (12 P.M.), the rate at which vehicles cross the bridge is decreasing at a rate of approximately 216 vehicles per hour per hour.</p>	<p>Approximation using values from table. 1 point 2.B</p> <hr/> <p>Interpretation with units 1 point 3.F 4.B</p>
Total for part A	
<p>B. Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\int_0^{12} R(t) dt$. Indicate units of measure.</p> $\int_0^{12} R(t) dt \approx 4(R(2) + R(6) + R(10))$ $= 4(3653 + 3010 + 1986)$ $= 34,596 \text{ vehicles}$	<p>Midpoint sum set up 1 point 1.E</p> <hr/> <p>Approximation using values from the table with units 1 point 2.B 4.B</p>
Total for part B	
<p>C. What is the average number of vehicles crossing the bridge per hour on the weekend day for $0 \leq t \leq 12$?</p> $\frac{1}{12 - 0} \int_0^{12} H(t) dt = 2452$ <p>Definite integral Answer</p> <p>integral</p>	<p>Definite integral 1 point 1.D 4.C</p> <hr/> <p>Answer with supporting work 1 point 1.E</p>
Total for part C	
<p>D. Use $L(t)$ to find the time t, for $12 \leq t \leq 17$, at which the rate of vehicles crossing the bridge is 2000 vehicles per hour. Show the work that leads to your answer.</p> $L(t) = H(12) - H'(12)(t - 12)$ $H(12) = 2596; H'(12) = -216$ $L(t) = 2000$ $\Rightarrow t = 14.759$	<p>Slope 1 point 1.E 4.E</p> <hr/> <p>$L(t) = 2000$ 1 point 1.D</p> <hr/> <p>Answer with supporting work 1 point 1.E 4.E</p>
Total for part D	
Total for Question 1	
9 points	

Scoring Guidelines for Question 2
Part B (AB or BC): Calculator Not Permitted
9 pointsGraph of f'

The figure above shows the graph of f' , the derivative of a twice-differentiable function f , on the closed interval $[0, 4]$. The areas of the regions bounded by the graph of f' and the x -axis on the intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, and $[3, 4]$ are 2, 6, 10, and 14, respectively. The graph of f' has horizontal tangents at $x = 0.6$, $x = 1.6$, $x = 2.5$, and $x = 3.5$. It is known that $f(2) = 5$.

- A. On what open intervals contained in $(0, 4)$ is the graph of f both decreasing and concave down? Give a reason for your answer.
- B. Find the absolute minimum value of f on the interval $[0, 4]$. Justify your answer.
- C. Evaluate $\int_0^4 f(x)f'(x)dx$.
- D. The function g is defined by $g(x) = x^3 f(x)$. Find $g'(2)$. Show the work that leads to your answer.

Model Solution		Scoring	
A.	On what open intervals contained in $(0,4)$ is the graph of f both decreasing and concave down? Give a reason for your answer.		
	The graph of f is decreasing and concave down on the intervals $(1, 1.6)$ and $(3, 3.5)$	Answer	1 point 2.E
	because f' is negative and decreasing on these intervals.	Reason	1 point 3.E 4.A
		Total for part A	2 points
B.	Find the absolute minimum value of f on the interval $[0, 4]$. Justify your answer.		
	The graph of f' changes from negative to positive only at $x = 2$.	Considers $x = 2$ as a candidate	1 point 3.B
	$f(0) = f(2) + \int_2^0 f'(x) \, dx = f(2) - \int_0^2 f'(x) \, dx = 5 - (2 - 6) = 9$ $f(2) = 5$ $f(4) = f(2) + \int_2^4 f'(x) \, dx = 5 + (10 - 14) = 1$ On the interval $[0, 4]$, the absolute minimum value of f is $f(4) = 1$.	Answer with justification	1 point 3.E
		Total for part B	2 points
C.	Evaluate $\int_0^4 f(x)f'(x) \, dx$.		
	$\int_0^4 f(x)f'(x) \, dx = \frac{1}{2}(f(x))^2 \Big _{x=0}^{x=4}$	Antiderivative of the form $a[f(x)]^2$	1 point 1.C
	$= \frac{1}{2}((f(4))^2 - (f(0))^2)$ $= \frac{1}{2}(1^2 - 9^2) = -40$	Earned the first point and $a = \frac{1}{2}$	1 point 1.E
		Answer	1 point 2.B
		Total for part C	3 points
D.	Find $g'(2)$. Show the work that leads to your answer.		
	$g'(x) = 3x^2f(x) + x^3f'(x)$ $g'(2) = 3 \cdot 2^2f(2) + 2^3f'(2) = 12 \cdot 5 + 8 \cdot 0 = 60$	Product Rule	1 point 1.E
		Answer	1 point 2.B
		Total for part D	2 points
		Total for Question 2	9 points

Scoring Guidelines for Question 3
Part A (BC only): Graphing Calculator Required**9 points**

For $0 \leq t \leq 5$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, -7)$. It is known that $\frac{dx}{dt} = \sin\left(\frac{t}{t+3}\right)$ and $\frac{dy}{dt} = e^{\cos t}$.

- A. Write an equation for the line tangent to the curve at the point $(2, -7)$.
- B. Find the y -coordinate of the position of the particle at time $t = 4$.
- C. Find the total distance traveled by the particle from time $t = 1$ to time $t = 4$.
- D. Find the time at which the speed of the particle is 2.5. Find the acceleration vector of the particle at this time.

Model Solution	Scoring						
<p>A. Write an equation for the line tangent to the curve at the point $(2, -7)$.</p> $\left. \frac{dy}{dx} \right _{t=1} = \left. \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right _{t=1} = \frac{e^{\cos 1}}{\sin\left(\frac{1}{4}\right)} = 6.938150$ <p>An equation for the line tangent to the curve at the point $(2, -7)$ is $y = -7 + 6.938(x - 2)$.</p>	<table> <tr> <td>Slope</td><td>1 point 1.C 4.E</td></tr> <tr> <td>Tangent line equation</td><td>1 point 1.D</td></tr> </table>	Slope	1 point 1.C 4.E	Tangent line equation	1 point 1.D		
Slope	1 point 1.C 4.E						
Tangent line equation	1 point 1.D						
Total for part A 2 points							
<p>B. Find the y-coordinate of the position of the particle at time $t = 4$.</p> $y(4) = -7 + \int_1^4 \frac{dy}{dt} dt = -5.006667$ <p>The y-coordinate of the position of the particle at time $t = 4$ is -5.007 (or -5.006).</p>	<table> <tr> <td>Definite integral</td><td>1 point 1.D 4.C</td></tr> <tr> <td>Answer</td><td>1 point 2.B</td></tr> </table>	Definite integral	1 point 1.D 4.C	Answer	1 point 2.B		
Definite integral	1 point 1.D 4.C						
Answer	1 point 2.B						
Total for part B 2 points							
<p>C. Find the total distance traveled by the particle from time $t = 1$ to time $t = 4$.</p> $\int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = 2.469242$ <p>The total distance traveled by the particle from time $t = 1$ to time $t = 4$ is 2.469.</p>	<table> <tr> <td>Definite integral</td><td>1 point 1.D 4.C</td></tr> <tr> <td>Answer</td><td>1 point 1.E 4.E</td></tr> </table>	Definite integral	1 point 1.D 4.C	Answer	1 point 1.E 4.E		
Definite integral	1 point 1.D 4.C						
Answer	1 point 1.E 4.E						
Total for part C 2 points							
<p>D. Find the time at which the speed of the particle is 2.5. Find the acceleration vector of the particle at this time.</p> $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2.5 \Rightarrow t = 0.415007$ <p>The speed of the particle is 2.5 at time $t = 0.415$.</p> <p>The acceleration vector of the particle at time $t = 0.415$ is $\langle x''(0.415), y''(0.415) \rangle = \langle 0.255, -1.007 \rangle$ (or $\langle 0.255, -1.006 \rangle$).</p>	<table> <tr> <td>Speed equation</td><td>1 point 1.D 4.C</td></tr> <tr> <td>Value of t</td><td>1 point 1.E 4.E</td></tr> <tr> <td>Acceleration vector</td><td>1 point 1.E 4.E</td></tr> </table>	Speed equation	1 point 1.D 4.C	Value of t	1 point 1.E 4.E	Acceleration vector	1 point 1.E 4.E
Speed equation	1 point 1.D 4.C						
Value of t	1 point 1.E 4.E						
Acceleration vector	1 point 1.E 4.E						
Total for part D 3 points							
Total for Question 3 9 points							

Scoring Guidelines for Question 4
Part B (BC only): Calculator Not Permitted**6 points**

The Maclaurin series for the function f is given by $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k^2} = x - \frac{x^2}{4} + \frac{x^3}{9} - \cdots$ on its interval of convergence.

- A. Use the ratio test to determine the interval of convergence of the Maclaurin series for f . Show the work that leads to your answer.
- B. The Maclaurin series for f evaluated at $x = \frac{1}{4}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $f\left(\frac{1}{4}\right)$ using the first two nonzero terms of this series is $\frac{15}{64}$. Show that this approximation differs from $f\left(\frac{1}{4}\right)$ by less than $\frac{1}{500}$.
- C. Let h be the function defined by $h(x) = \int_0^x f(t) dt$. Write the first three nonzero terms and the general term of the Maclaurin series for h .

Model Solution	Scoring
<p>A. Use the ratio test to determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.</p> $\lim_{k \rightarrow \infty} \left \frac{(-1)^{k+2} x^{k+1}}{(k+1)^2} \cdot \frac{(k+1)^2}{(-1)^{k+1} x^k} \right = \lim_{k \rightarrow \infty} \frac{k^2}{(k+1)^2} x = x $ <p>$x < 1$</p> <p>The series converges for $-1 < x < 1$.</p> <p>When $x = -1$, the series is $\sum_{k=1}^{\infty} \frac{-1}{k^2}$. This is a convergent p-series.</p> <p>When $x = 1$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$. This series converges by the alternating series test.</p> <p>The interval of convergence of the Maclaurin series for f is $-1 \leq x \leq 1$.</p>	<p>Sets up ratio 1 point 3.B</p> <hr/> <p>Computes limit of ratio 1 point 1.E 4.C</p> <hr/> <p>Identifies interior or interval of convergence 1 point 3.D</p> <hr/> <p>Considers both endpoints 1 point 1.D</p> <hr/> <p>Analysis and interval of convergence 1 point 3.D</p>
<p style="text-align: right;">Total for part A 5 points</p>	
<p>B. Show that this approximation differs from $f\left(\frac{1}{4}\right)$ by less than $\frac{1}{500}$.</p> $\left f\left(\frac{1}{4}\right) - \frac{15}{64} \right < \frac{\left(\frac{1}{4}\right)^3}{9} = \frac{1}{576}$ $\frac{1}{576} < \frac{1}{500}$	<p>Uses third term as error bound 1 point 3.D</p> <hr/> <p>Error bound 1 point 3.E</p> <p style="text-align: right;">Total for part B 2 points</p>
<p>C. Write the first three nonzero terms and the general term of the Maclaurin series for h.</p> $h(x) = \int_0^x f(t) dt = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{36} - \dots + \frac{(-1)^{k+1} x^{k+1}}{(k+1)k^2} + \dots$ <p style="text-align: center;"> First three nonzero terms General term </p>	<p>First three nonzero terms 1 point 1.D</p> <hr/> <p>General term 1 point 1.D 4.C</p> <p style="text-align: right;">Total for part C 2 points</p>
<p style="text-align: right;">Total for Question 4 9 points</p>	

